GYANASRA

VOL.20 1990

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ISSN 0304-9892

Jñānābha ज्ञानाभ

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पुस्तकालय
गु० कां० विश्वविद्यालय
हरिद्वार

3/10-90

VOLUME 20

1990

Published by
THE VIJNANA PARISHAD OF INDIA
DAYANAND VEDIC POSTGRADUATE COLLEGE
(Bundelkhand University)
ORAI, U. P., INDIA

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Jñānabha, Vol. 20, 1990

Iterative construction of solutions of functional equations involving multivalued operators in L^P spaces

by

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(Received : July 6, 1989)

INTRODUCTION

The aim of this paper is to give an iterative method converging strongly to a solution of the equation $f \in x + T(x)$ or to a solution of the equation $x \in T(x)$ in L_P spaces, $p \geqq 2$, where T is a multivalued monotone operator from L^p into itself.

The main tool we will use is a technique similar to that used by Dunn [6] and Dotson [5] for the convergence to a fixed point of a multivalued monotone operator defined in a Hilbert space or to a solution of the equation $f = x + T(x)$, respectively.

To do this we will make use of a result due to Bynum [2] and another due to Dunn [6].

For other results in the direction of this paper see also Dotson [5], Bruck [1], Chidume [3], [4].

NOTATIONS DEFINITIONS AND PRELIMINARY RESULTS

Let X be a real Banach space with norm $\| - \|$ and X^* its dual : denote by $<\cdot, \cdot>$ the generalized duality pairing.

Denote by $J : X \to X^*$ the (multivalued) duality map given by

$$J(x) = \{ f^* \in X^* : \| f^* \|^2 = \| x \|^2 = <x, f^*> \}.$$

If X^* is strictly convex, then J is single-valued.

Let T be an operator with domain $D(T)$ and range $R(T)$ in X. T is said to be **monotone** if

$$\| x - y \| \leqq \| x - y + t(\xi - n) \| \quad ..(1)$$

for every $x, y \in D(T)$, $\xi \in T(x)$, $\eta \in T(y)$ and some $t > 0$ (see, *e.g.* Kato [7]).

We can define (1) in terms of the duality map J as follows (see, *e.g.* Kato [7]) :

$$Re < \xi - n, j > \geqq 0 \quad ...(2)$$

for every $x, y \in D(T)$, $\xi \in T(x)$, $n \in T(y)$ and for some $j \in J(x - y)$.

If X is a Hilbert space, then condition (2) becomes

$$Re < \xi - n,\ x - y \geqq 0$$

for every $x, y \in D(T)$, $\xi \in T(x)$, $n \in T(y)$.

An operator $T : D(T) \to R(T)$ is said to be **bounded** if there exists $d > 0$ such that, for every $x, y \in D(T)$, for every $\xi \in T(x)$, $n \in T(y)$, we have

$$\| \xi - n \| \leqq d.$$

The following results are crucial in the proofs of our theorems.

Lemma 1. (see Bynum [2]). Let $X = L^p$, $p \geqq 2$. Then for every $x, y \in x$, we have

$$\| x + y \|^2 \leqq (p-1) \ \| x \|^2 + \| y \|^2 + 2 < x, J(x) > .$$

Lemma 2. (see Dunn [6]). Let β_n be recursively generated by

$$\beta_{n+1} = (1 - \delta_n)\beta_n + \sigma_n^2.$$

with $n \geqslant 1$, $\beta_1 \geqq 0$, $\{\delta_n\} \subset [0, 1]$ and

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty \qquad \sum_{n=1}^{\infty} \delta_n = \infty.$$

Then $\beta_n \geqq 0$ for $n \geqq 1$ and $\lim_{n \to \infty} \beta_n = 0$.

RESULTS

We prove the following

Theorem 1. Let $T : L^p \to L^p$, $p \geqq 2$, be a multivalued monotone bounded operator. Assume that the equation $f \in x + T(x)$, $f \in L^p$, has a solution $\bar{x} \in L^p$, where $f \in L^p$.

Define $S : L^p \to L^p$ as follows :

$$S(x) = -T(x) + f.$$

Let $\{t_n\} \subset R$ be a sequence satisfying

$$0 \leqq t_n \leqq 1, \quad \sum_{n=1}^{\infty} t_n = \infty, \quad \sum_{n=1}^{\infty} t_n^2 < \infty.$$

Then the sequence $\{x_n\} \subset L^p$, generated by

$$x_{n+1} = (1 - t_n)x_n + t_n \xi_n$$

$\xi_n \in S\ x_n)$, $x_1 \in L_p$, converges strongly to the solution $\bar{x}$ of $f \in x + T(x)$.

Proof. Let $\bar{x}$ be a solution of the equation $f \in x + T(x)$, hence $\bar{x} \in S(\bar{x})$. The boundedness of T implies that there exists a $k > 0$ such that

$$\| n - \delta \| \leqslant k \qquad \ldots(3)$$

for every $x, y \in L^p$, $n \in S(x)$ and $\delta \in S(y)$.

Furthermore, from the monotonicity of T, we obtain that

$$<n-\delta,\ J(x-y)> \leqslant 0 \qquad \text{...(4)}$$

for all $x,\ y \in L^p,\ n \in S(x),\ \delta \in S(y)$.

Let $\xi_n \in S(x_n)$ and $\bar{\xi} \in S(\bar{x})$. From Lemma 1 we get

$$\| x_{n+1} - \bar{x} \|^2 = \| (1-t_n)(x_n - \bar{x}) + t_n(\xi_n - \bar{\xi}) \|^2$$

$$\leqq (p-1)t_n^2 \| \xi_n - \bar{\xi} \|^2 + (1+t_n)^2 \| x_n - \bar{x} \|^2$$

$$+ 2t_n(1-t_n) < \xi_n - \bar{\xi},\ J(x_n - \bar{x}) > .$$

Since $0 \leqq t_n \leqslant 1$, we have

$$(1-t_n)^2 \leqq (1-t_n). \qquad \text{...(5)}$$

Taking in account the inequalities (3), (4) and (5), we obtain

$$\| x_{n+1} - \bar{x} \|^2 \leqslant (p-1)t_n^2 k^2 + (1-t_n) \| x_n - \bar{x} \|^2 \qquad \text{...(6)}$$

Put

$$\| x_n - \bar{x} \|^2 = r_n.$$

Then we can rewrite (6) in the form

$$r_{n+1} \leqq (1-t_n) r_n + Q^2 t_n^2,$$

where $Q^2 = (p-1)k^2$.

Let $\{a_n\} \subset R$ be defined by

$$a_{n+1} = (1-t_n)a_n + t_n^2,\ a_1 = 1.$$

We have

$$0 \leqq r_n \leqq A^2\ a_n, \qquad \text{...(7)}$$

where

$$A^2 = \max\{r_1,\ Q^2\}. \qquad \text{...(8)}$$

We prove (7) by induction; namely, if $n=1$ we have, by (8),

$$r_1 \leqq A^2 a_1 = A^2.$$

Assume that (7) holds.

Clearly $r_{n+1} \geqslant 0$; moreover

$$r_{n+1} \leqq (1-t_n)r_n + O^2 t_n^2 \leqq (1-t_n)A^2 a_n + A^2 t_n^2 = A^2 a_{n+1}.$$

Put $a_n = \beta_n$, $\delta_n = t_n = \sigma_n$ for each $n \in N$. From Lemma 2, it follows that $a_n \to 0$, as $n \to \infty$, so that, from (7) it follows that $r_n \to 0$, as $n \to \infty$.

Hence $\{x_n\}$ converges strongly to x.

REMARK

A particular case is of some interest since it gives an additional information of an error estimate.

Assume that

$$t_n = \frac{1}{n+1}.$$

We want to show that

$$\| x_n - \bar{x} \| = 0\left(\frac{1}{\sqrt{n}} \right). \qquad \text{.. (9)}$$

In fact, by (6) of Theorem 1, we have

$$\| x_{n+1} - \bar{x} \|^2 \leqq Q^2 t_n^2 + (1-t_n)^2 \| x_n - \bar{x} \|^2. \qquad \text{...(10)}$$

Recall that

$$A^2 = \max \{ r_1,\ Q^2 \}$$

and put

$$d_n = \frac{A^2}{n}.$$

To obtain (9) it sufficies to show that

$$\| x_n - \bar{x} \|^2 \leqq d_n^2. \qquad \text{...(11)}$$

We prove (11) by induction.

For $n=1$ we obtain

$$\| x_1 - \bar{x} \|^2 = r_1 \leqq A^2.$$

Assume that (11) is true. From (10) we obtain that

$$\| x_{n+1} - \bar{x} \|^2 \leqq Q^2 t_n^2 + (1-t_n)^2 d_n^2.$$

Since

$$t_n = \frac{1}{n+1}$$

we have

$$\| x_{n+1} - \bar{x} \|^2 \leqq \frac{Q^2}{(n+1)^2} + \frac{n^2}{(n+1)^2}\, \frac{A^2}{n} \leqslant \frac{A^2}{(n+1)^2}\,(1+n) = d_{n+1}^2$$

In the next theorem we construct a sequence converging strongly to a fixed point of a multivalued monotone operator T.

We can find a similar result in Dunn [6] for operators acting in Hilbert spaces.

Theorem 2. Let

$$B_r = \{ x \in L^p :\ \| x \| \leqslant r \}.$$

and let $T : B_r \to B_r$ be a multivalued monotone operator.

Assume that T has a fixed point $\bar{x} \in B_r$. Let $\{t_n\} \subset [0,\ 1]$ be such that

$$\sum_{n=1}^{\infty} t_n^2 < \infty \text{ and } \sum_{n=1}^{\infty} t_n = \infty.$$

Then the sequence recursively generated by

$$x_{n+1}=(1-t_n)x_n+t_n\xi_n,$$

$\xi_n\in -T(x_n)$, $x_1\in B_r$, converges strongly to $\bar{x}$.

Proof. We have, for every $n\in N$.

$$x_{n+1}\in B_r.$$

In fact $x_1\in B_r$. If $x_n\in B_r$ then

$$\| x_{n+1} \| \leqq (1-t_n)\| x_n \| + t_n \| \xi_n \| \leqq (1-t_n)r+t_nr=r.$$

The remaining part of the proof runs as the proof of Theorem 1.

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Jñānābha, Vol, 20, 1990

An Extension of the Bernoulli and Euler Polynomials

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(Received : September 13, 1985, Revised : August 8, 1988)

ABSTRACT

The present paper studies a general class of polynomials which unify the Bernoulli and Euler polynomials together with some of their recent generalizations.

1. INTRODUCTION

The study of Bernoulli and Euler polynomials has contributed much to our knowledge of the theory of numbers. In recent years, the Eulerian numbers and certain generalizations have been encountered in a number of combinatorial problems. Keeping these recent developments, we study a polynomial set which unifies the earlier studies on the subject by, for example, Rai and Singh [4] and Srivastava [6].

Let a, b, c be non-zero real numbers and let k, p, q, μ, ν be positive integers. Define the polynomials $\lambda_n \ (x, y : a, b, c, k, p, q, \mu, \nu)$ by the generating relation :

$$\frac{c^2(t^3/c^2)^k \exp(x^\mu t^p + y^\nu t^q)}{[\exp(t^p)-a]\,[\exp(t^q)-b]} = \sum_{n=0}^{\infty} \lambda_n \ (x, y : a, b, c, k, p, q, \mu, \nu)\frac{t^n}{n!} \tag{1.1}$$

The special case of (1.1) when $\mu=\nu=p=q-1=1$ yields the relationship

$$\lambda_n(x, y; a, b, c, k, 1, 2, 1, 1) = A_n(x, y; a. b, c, k), \tag{1.2}$$

where the polynomials $A_n(x, y; a, b, c, k)$ were studied by Srivastava [6, pp. 150-163] and contain, as particular cases, the various earlier generalizations of the Bernoulli and Euler polynomials (and numbers) due, for example, to Carlitz [1], and Rai and Singh [4]. See also Srivastava and Manocha [7, pp. 85 and 397]. Thus each of our results listed in the next sections would apply, under suitable special cases, to each of these systems of polynomials and numbers.

2. MAIN RESULTS

(*i*) Consider

$$\frac{\exp(xt+yt^2)}{\left[\exp(t)-\frac{1}{\xi}\right]\left[\exp(t)-\frac{1}{\eta}\right]}=\frac{\xi\eta(1-\xi)\exp(xt)(1-\eta)\exp(yt^2)}{(1-\xi)(1-\eta)[1-\xi\exp(t)][1-\eta\exp(t^2)]}$$

from which we get

$$\lambda_n\left((x,y;\frac{1}{\xi},\frac{1}{\eta},1,0,1,2,1,1\right)$$

$$=\frac{\xi\eta}{(1-\xi)(1-\eta)}\sum_{r=0}^{[n/2]}\frac{n!}{r!(n-2r)!}\phi_{n-2r}(x,\xi)\phi_r(y,\eta)$$

where $\phi_n(x,\xi)$ are defined by Carlitz.

(*ii*) We have, $\lambda_n(x,y;a,b,2,k,1,2,1,1)$

$$=n!\sum_{r=0}^{[n/2]}\frac{D_{n-2r}(x;a,k)\,D_r(y;b,k)}{r!(n-2r)!}$$

where $D_n(x;a,k)$ are the polynomials introduced by Karande and Thakare [3].

(*iii*) GENOCHI NUMBER

Putting $-a=-b=c=k=p=\mu=\nu=q-1=1$ and $x=0=y$ in (1.1), we obtain

$$\lambda_n(0\cdot 0;-1,-1,1,1,1,2,1,1,)=n!\sum_{r=0}^{[n/2]}\frac{G_{n-2r}}{r!(n-2r)!}G_r.$$

(*iv*) TWO MULTIPLICATION THEOREMS

From (1.1) we observe that

$$\sum_{n=0}^{\infty}\lambda_n(m^px,m^qy;a,b,c,k,p,q,\mu,\nu)\frac{t^n}{n!}$$

$$=c^2(t^3/c^2)^k\exp[m^px^\mu t^p+m^qy^\nu t^q].$$

$$\frac{\sum_{r=0}^{m^p-1}\left(\frac{1}{a}\right)^r\exp(rt^p)\sum_{s=0}^{m^q-1}\left(\frac{1}{b}\right)^s\exp(st^q)}{a[1-a^{-m^p}\exp(m^pt^p)]\,b[1-b^{-m^q}\exp(m^pt^q)]},\ (m=1,2,3,\ldots\ldots).$$

Now, using summation formula for a finite geometric series, the right-hand side becomes

$$\frac{a^{m^p-1}\, b^{m^q-1}}{m^{3\kappa}} \sum_{n=0}^{\infty} \sum_{n=0}^{m^p-1} \sum_{s=0}^{m^q-1} a^{-r} b^{-s} \lambda_n\Big[x+\frac{r}{m^p},$$

$$y+\frac{s}{m^q};\; a^{m^p},\; b^{m^q},\; c, k, p, q, \mu, \nu \Big] \cdot \frac{(mt)^n}{n!}.$$

Comparing the coefficients of t^n from both sides, we prove THEOREM 1 :

$$\lambda_n(m^p x,\, m^q y;\, a, b, c, k, p, q, \mu, \nu) = a^{m^p-1}\, b^{m^q-1}\, m^{n-3k}$$

$$\sum_{r=0}^{m^p-1} \sum_{s=0}^{m^q-1} a^{-r} b^{-s} \lambda_n\Big[x+\frac{r}{m^p},\, y+\frac{s}{m^q},\, a^{m^p},\, b^{m^q},\, c, k, p, q, \mu, \nu \Big]$$

Again, we consider

$$\sum_{n=0}^{\infty} \sum_{r=0}^{m^p-1} \sum_{s=0}^{m^q-1} a^{-r} b^{-s} \lambda_n \Big(X+\frac{r}{m^p},\; y+\frac{s}{m^q};\; a^{m^p},\, b^{m^q},$$

$$c, k, p, q, \mu, \nu \Big) \frac{(mt)^n}{n!}$$

$$= \sum_{r=0}^{m^p-1} \sum_{s=0}^{m^q-1} \frac{c^2 (m^3 t^3/c^2)^k \exp\Big[\Big(x^\mu+\frac{r}{m^p}\Big)(mt)^p + \Big(y^\nu+\frac{s}{m^q}\Big)(mt)^q\Big]}{a^r b^s\, [\exp(mt)^p - a^{m^p}][\exp(mt)^q - b^{m^q}]}$$

Now, simplifying and compairing the coefficients of t^n, we have

$$\frac{(n+3)!}{c^2\, n!} a^{1-m^p}\, b^{1-m^q}\, m^{3k}\, \lambda_n(m^p x,\, m^q y\, ;\, a, b, c, k-1, p, q, \mu, \nu)$$

$$= \sum_{r=0}^{m^p-1} \sum_{s=0}^{m^q-1} a^{-r} b^{-s} m^{n+3} \lambda_{n+3}\Big[x+\frac{r}{m^p},\; y+\frac{s}{m^q};\; a^{m^p},\, b^{m^q},$$

$$c, k, p, q, \mu, \nu \Big]$$

which is the THEOREM 2.

The special cases of the above theorems when $\mu=\nu=p=q-1=1$ yield the multiplication theorems given by Srivastava [6, *pp.* 150–163] and contain, as particular cases, the generalized Euler and Bernoulli polynomials.

3. SYSTEM OF PARTIAL DIFFRENTIAL EQUATIONS

Differentiating equation (1.1) with respect to x and equating coefficients of t^n, we have

(3.1) $$\frac{1}{n!} \frac{\partial \lambda_n}{\partial x} (x, y;\, a, b, c, k, p, q, \mu, \nu)$$

$$= \frac{\mu x^{\mu-1}}{(n-p)!} \lambda_{n-p}(x, y;\, a, b, c, k, p, q, \mu, \nu).$$

Similarly, we obtain

$$(3.2)\qquad \frac{1}{n!}\,\frac{\partial^2\lambda_n}{\partial x^2}(x, y; a, b, c, k, p, q, \mu, \nu)$$

$$=\frac{\mu^2}{(n-2p)!}\,\lambda_{n-2p}(x, y; a, b, c, k, p, q, \mu, \nu)\,x^{2(\mu-1)}$$

$$+\frac{\mu(\mu-1)x^{\mu-2}}{(n-p)!}\,\lambda_{n-p}(x, y; a, b, c, k, p, q, \mu, \nu)$$

and

$$(3.3)\qquad \frac{1}{n!}\,\frac{\partial^h\lambda_n}{\partial x^h}(x, y; a, b, c, k, p, q, \mu, \nu)$$

$$=\lambda_{n-hp}(x, y; a, b, c, k, p, q, \mu, \nu)\,\frac{\mu^h\, x^{h(\mu-1)}}{(n-hp)!}$$

$$+\lambda_{n-(h-1)p}(x, y; a, b, c, k, p, q, \mu, \nu)\; h\mu^{h-1}[\mu-\{h-(h-1)\}]$$

$$\frac{x^{(h-1)\mu-h}}{\{n-(h-1)p\}!}+\dots+\lambda_{n-\{h-(h-1)p\}}(x, y; a, b, c, k, p, q, \mu, \nu)$$

$$\frac{\mu(\mu-1)\dots\{\mu-(h-1)\}}{[n-\{h-(h-1)p\}]!}.$$

Again, differentiating equation (1.1) with respect to y, we get

$$(3.4)\qquad \frac{1}{n!}\,\frac{\partial\lambda_n}{\partial y}(x, y; a, b, c, k, p, q, \mu, \nu)$$

$$=\lambda_{n-q}(x, y; a, b, c, k, p, q, \mu, \nu)\;\frac{\nu y^{\nu-1}}{(n-q)!}.$$

Putting $\mu=\nu=p=q-1=1$ in (3.2) and (3.4), we obtain

$$(3.5)\qquad \frac{\partial\lambda_n}{\partial y}(x, y; a, b, c, k, 1, 2, 1, 1)=\frac{\partial^2\lambda_n}{\partial x^2}(x, y; a, b, c, k, 1, 2, 1, 1)$$

from which we observe that all functions $\lambda_n(x, y; a, b, c, k, 1, 2, 1, 1)$ are solutions of one-dimentional heat diffusion equations.

Taking $\mu=1$, $x=x+h$ in equation (1.1) and equating coefficients of t^n, we have

$$(3.6)\qquad \frac{1}{n!}\,\lambda_n(x+h, y; a, b, c, k, p, q, 1, \nu)$$

$$=\sum_{r=0}^{[n/p]}\frac{\lambda_{n-pr}(x, y; a, b, c, k, p, q, 1, \nu)}{r!\,(n-pr)!}.$$

Writing $\mu=1$ and $1-x$ for x in (1.1), we obtain

$$(3.7)\qquad \lambda_n(1-x, y; a, b, c, k, p, q, 1, \nu)$$

$$=\frac{(-1)^{n+k+1}}{a}\,\lambda_n\Big[(-1)^{p+1}x, (-1)^q y; \frac{1}{a}, b, c, k, p, q, -1, \nu\Big].$$

Replacing x by $1+x$ in (3.7), we obtain

$$a\,\lambda_n\,(-x,\,y;\,a,\,b,\,c,\,k,\,p,\,q,\,1,\,\nu) \tag{3.8}$$

$$=(-1)^{n+k+1}\,\lambda_n\,[(-1)^{p+1}\,(1+x),\,(-1)^q y;\,\frac{1}{a},\,b,\,c,\,k,\,p,\,q,\,1,\,\nu].$$

Now, for $\mu=1$, we can easily obtain from (1.1)

$$a\,\lambda_n\,(-x,\,y;\,a,\,b,\,c,\,k,\,p,\,q,\,1,\,\nu) \tag{3.9}$$

$$=(-1)^{n+k+1}\sum_{r=0}^{n}(-1)^{r-pr}\binom{n}{r}\lambda_{n-pr}\Big[(-)^{p+1}\,x,\,(-1)^q\,y;\frac{1}{a},$$

$$b,\,c,\,k,\,p,\,q,\,1,\,\nu\Big].$$

Eliminating $a\,\lambda_n\,(-x,\,y;\,a,\,b,\,c,\,k,\,p,\,q,\,1,\,\nu)$ from (3.8) and (3.9), we arrive at

$$\lambda_n\Big[(-1)^{p+1}\,x,\,(-1)^q\,y;\frac{1}{a},\,b,\,c,\,k,\,p,\,q,\,1,\,\nu\Big] \tag{3.10}$$

$$=\sum_{r=0}^{n}(-1)^{r(1-p)}\binom{n}{r}\lambda_{n-pr}\Big[(-1)^{p+1}\,x,\,(-1)^q y;\frac{1}{a},$$

$$b,\,c,\,k,\,p,\,q,\,1,\,\nu\Big].$$

The particular cases of the above results when $p=q-1=\nu=1$ yield the respective results given by Srivastava [6]; for example,

$$\lambda_n\,(x+h,\,y;\,a,\,b,\,c,\,k,\,1,\,2,\,1,\,1)=A_n(x+h,\,y;\,a,\,b,\,c,\,k).$$

ACKNOWLEDGEMENTS

The authors are extremely grateful to Prof. H. M. Srivastava for suggesting a number of improvements in this paper.

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Jñānabha, Vol, 20, 1990

A New Starting Point for The Hermite Polynomials

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(Received : April 6, 1988, Revised [Final] : April 6, 1989)

ABSTRACT

We show that the beautiful relation

$$H_n(x)=(2x-D)^n.\ 1,\ D\equiv d/dx,\ n\geqslant 0,$$

is a new starting point for developing the properties of the Hermite polynomials by obtaining from it all the standard properties of $H_n(x)$.

1. INTRODUCTION

It is customary to obtain the properties of $H_n(x)$, the Hermite polynomials [5], starting either at the Hermite differential equation (*e.g.*, [4], or at the generating function (*e.g.*, [1], [2], [3]). It is also possible to deduce the properties of $H_n(x)$, $n\geqslant 0$, starting from the Rodrigues representation [8]. The purpose of this article is to show that the interesting and simple relation [2, problem 13. 1. 5, p, 718]

$$H_n(x)=2x-D)^n,\ 1,\ D\equiv d/dx,\ n\geqslant 0 \qquad ...(1)$$

is a *new* starting point for devloping the properties of the Hermite polynomials [5].

2. SIMPLE CONSEQUENCES OF EQUATION (1)

Throughout this article, we follow the convention that $W^0=I$, the unit operator, for any operator W. Thus $H_0(x)=1$. Let

$$B\equiv 2x-D. \qquad ...(2)$$

Since

$$B^{n+1}=BB^n,\ n\geqslant 0,$$

$$H_{n+1}(x)=(2x-D)H_n(x), \quad n\geqslant 0, \qquad ...(3a)$$

$$H_m(x)=(2x-D)H_{m-1}(x),\ m\geqslant 1. \qquad ...(3b)$$

It follows from equations (3) and induction that the leading term of $H_n(x)$ is $2^n x^n$ and hence $H_n(x)$ is a polynomial of precise degree n in x. From (1), $H_n(-x)=(-1)^n H_n(x)$, $n\geqslant 0$, since $D\rightarrow -D$ when $x\rightarrow -x$. We say that $H_n(x)$ has a definite parity $(-1)^n$, *i.e.* the Hermite polynomials $H_n(x)$ are even or odd polynomials in x according to whether n is even or odd. Therefore,

$$H_{2M+1}(0)=0,\ M\geqslant 0. \quad \ldots(4)$$

3. ORTHONORMALITY PROPERTY

We shall first prove the orthonormality property. Let

$$I_{mn}=\int_{-\infty}^{\infty} \exp(-x^2)H_m(x)H_n(x)dx. \quad \ldots(5)$$

Using (3*b*), we have, $\exp(-x^2)H_m(x)=-D\{\exp(-x^2)H_{m-1}(x)\}$, $m\geqslant 1$. Integration by parts and iteration yield

$$I_{mn}=\int_{-\infty}^{\infty} \exp(-x^2)D^m\{H_n(x)\}dx. \quad (6a)$$

(Note that (6*a*) is also true when $m=0$.) As $I_{mn}=I_{nm}$' symmetry demands

$$I_{mn}=\int_{-\infty}^{\infty} \exp(-x^2)D^n\{H_m(x)\}dx. \quad \ldots(6b)$$

If $m>n$, $D^m\{H_n(x)\}\equiv 0$; if $n>m$, $D^n\{H_m(x)\}\equiv 0$. Hence, from (6*a*) and (6*b*),

$$I_{mn}=0,\ m\neq n. \quad \ldots(7)$$

If $m=n$, $D^m\{H_n(x)\}=2^n n!$, since the leading term of $H_n(x)$ is $2^n x^n$. Hence, from (6*a*) or (6*b*),

$$I_{nn}2^n n!\int_{-\infty}^{\infty} \exp(-x^2)dx=2^n n!\sqrt{\pi} \quad \ldots(8)$$

The orthonormality property

$$\int_{-\infty}^{\infty} \exp(-x^2)H_m(x)H_n(x)dx=2^n n!\sqrt{\pi}\ \delta_{mn}, \quad \ldots(9)$$

follows from (7) and (8). We shall now use it to prove the recurrence relations.

4. RECURRENCE RELATIONS

The leading term of $H_m(x)$ is $2^m x^m$ and $DH_n(x)$ is a polynomial of degree $n-1, n\geqslant 1$. Moreover, $H_p(x)$ has a definite parity $(-1)^p$. Since any real polynomial in x is a linear combination of the Hermite polynomials [5], we have

$$DH_n(x)=2nH_{n-1}(x)+f(x), \quad \ldots(10)$$

where $f(x)\equiv 0$ if $n=1, 2$; for $n\geqslant 3$,

$$f(x)=\sum_{r=1}^{[(n-1)/2]} C_{n-1-2r}\, H_{n-1-2r}(x), \quad \ldots(11)$$

where $[t]$ is the greatest integer $\leqslant t$. Equations (3*a*) and (10) imply that

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)-f(x),\ n\geqslant 1. \quad \ldots(12)$$

Now $H_N(x)$, $N>0$, will be orthogonal to every polynomial of lower degree with respect to the weight function $\exp(-x^2)$, in the interval $[-\infty, \infty]$. Note that $xH_{n-1-2r}(x)$ is a polynomial of precise degree $n-2r$ and that $0\leqslant n-1-2r>n-2r<n-1<n<n+1$, $n\geqslant 3$. Hence, with respect to $\exp(-x^2)$ and in the interval $[-\infty,\infty]$, $H_{n+1}(x)$ and $H_{n-1}(x)$ are orthogonal to $H_{n-1-2s}(x)$

and $H_n(x)$ is orthogonal to $xH_{n-1-2s}(x)$, $1\leqslant s\leqslant[(n-1)/2]$, $n\geqslant 3$. Therefore, from (9), (11), and (12), $C_{n-1-2s}\equiv 0$, for all allowed values of s. Thus $f(x)\equiv 0$ for all $n\geqslant 1$. From (10)–(12), one obtains the recurrence relations :

$$H_{n+1}(x) = 2n\,H_n(x) - 2x\,H_{n-1}(x),\ n\geqslant 1, \qquad \text{...(13)}$$

$$DH_n(x) = 2n\,H_{n-1}(x),\ n\geqslant 1. \qquad \text{...(14)}$$

An independent and sophisticated way of obtaining the differential recurrence relation, equation (14), involves the use of commutator brackets [2]. These powerful mathematical tools are essential to express the beauty and elegance of certain laws of Quantum Physics. The commutator of the operators S and T, denoted by $[S, T]$, is defined by $[S, T]=ST-TS$. Now $DH_n(x)=DB^n\cdot 1=\{[D, B^n]+B^n D\}\cdot 1=[D, B^n]\cdot 1$, as $B^n D\cdot 1\equiv 0$, and $[P, Q]R+Q[P, R]=\{PQ-QP\}R+Q\{PR-RP\}=P(QR)-(QR)P=[P, QR]$. Moreover, $[D, B]=[D, 2x-D]=[D, 2x]-[D, D]=[D, 2x]$, as any operator commutes with itself. Since $[D, 2x]g(x)=D(2xg)-2x(Dg)=2g$, for *any* differentiable function $g(x)$, we have the following results :

$$[P, QR]=[P, Q]R+Q[P, R],\ [D, B]=2B^0, \qquad \text{...(15)}$$

$$DH_n(x)=[D, B^n]\cdot 1,\ n\geqslant 1. \qquad \text{...(16)}$$

Since $B^{m+1}=B\,B^m$ and $[D, B^{m+1}]=[D, B]\,B^m+B\,[D, B^m]$, $m\geqslant 0$, it follows from (15), (16) and induction that

$$[D, B^n]=2n\,B^{n-1},\ n\geqslant 1. \qquad \text{...(17)}$$

The differential recurrence relation follows immediately from (16) and (17). From (14) and (3*a*), one can obtain the three-term recurrence relation, equation (13).

5. DIFFERENTIAL EQUATION

While the three-term recurrence relation involves successive Hermite polynomials, the differential recurrence relation is a first order differential equation, but with polynomials of two adjacent indices. One may wonder what price one has to pay to have all indices alike : No derivative, three different indices (see (13)); first derivative, two different indices (see (14)). Perhaps second derivative, single index !?? The answer is a second order linear differential equation which we are now going to derive. From (14) and (3*b*), we have, $D^2H_n(x)=DDH_n(x)=D\{2n\ H_{n-1}(x)\}=2n\,\{2x\,H_{n-1}(x)-H_n(x)\}=2x\,DH_n(x)-2n\,H_n(x)$, $n\geqslant 1$. However, this result is also true for $n=0$. Hence the differential equetion satisfied by the Hermite polynomials is given by

$$(D^2-2x\,D+2n)\,H_n(x)=0,\ n\geqslant 0. \qquad \text{...(18)}$$

6. RODRIGUES' REPRESENTATION

Using (3*b*), we have, $\exp(-x^2)H_m(x)=(-D)\{\exp(-x^2)H_{m-1}(x)\}$, $m\geqslant 1$.

Interation yields $\exp(-x^2)\, H_m(x) = (-D)^m \{\exp(-x^2)\, H_0(x)\}$, $m \geqslant 1$. This result, however, is also true for $m=0$. It follows that the Rodrigues' representation of the Hermite polynomials is given by

$$H_n(x) = (-1)^n \exp(x^2)\, D^n \{\exp(-x^2)\},\ n \geqslant 0, \qquad \dots(19)$$

since $H_0(x)=1$. (As the operator relation $F(-D)^n \equiv B^n F$ exists, F being simply $\exp(x^2)\, B^0$, $n \geqslant 0$, (1) and (19) are really equivalent.)

7. POWER SERIES EXPANSION

We can obtain, using (3*b*) and induction, the power series expansion of $H_n(x)$ [5],

$$H_n(x) = \sum_{s=0}^{[n/2]} \frac{(-1)^s\, n!\, (2x)^{n-2s}}{s!\, (n-2s)!},\ n \geqslant 0. \qquad \dots(20)$$

Although the method is straightforward, it is a bit tedious. For an alternative proof of arriving at (20) from (1), see also [8].

8. GENERATING FUNCTION

It follows from (13) and induction (or iteration) that $H_{2m}(0) = (-1)^m (2m)!/m!$; moreover (see (4)), $H_{2m+1}(0) = 0$, $m \geqslant 0$. One can now follow Arfken [2, Problem 13·1·1, *p.* 717 to find the generating function

$$\exp(2tx - t^2) = \sum_{n=0}^{\infty} H_n(x)\, t^n/n! \qquad \dots(21)$$

Following Simmons [6, *p.* 191], it is possible to obtain the generating function from the power series expansion also. See below for an alternative proof.

9. IDENTITIES INVOLVING THE HERMITE POLYNOMIALS

We shall now exhibit the usefulness of (1) in proving the following well-known identities involving the Hermite polynomials :

(*a*)
$$S(k, x, t) = \sum_{n=0}^{\infty} H_{n+k}(x)\, t^n/n!$$
$$= \exp(2tx - t^2)\, H_k(x-t),\ k \geqslant 0. \qquad \dots(22)$$

(*b*)
$$H_n(x) = 2^n \exp(-D^2/4)\, x^n,\ n \geqslant 0. \qquad \dots(23)$$

While (22) is from Rainville (5, *p.* 179], (23) is from Bell [3, *Th.* 5.3, *p.* 159]. First we consider (22). Using (1), we have

$$S(k, x, t) = \sum_{n=0}^{\infty} (1/n!)\, t^n\, (2x-D)^{n+k} \cdot 1$$
$$= \exp\{t(2x-D)\}\, (2x-D)^k \cdot 1. \qquad \dots(24)$$

Now [10, *p*. 966] $\exp(A+B)=\exp(A)\exp(B)\exp(-C/2)$, if $[A, C]=0=[B, C]$, with $C=[A, B]$. If we choose $A=2tx$ and $B=-tD$, then $C=2t^2$. Hence $[A, C]=0=[B, C]$. Therefore, $\exp(2tx-tD)=\exp(2tx)\exp(-tD)\exp(-t^2)=\exp(2tx)\exp(-t^2)\exp(-tD)$, since $\exp(-tD)$ and $\exp(-t^2)$ commute. Thus

$$S(k, x, t)=\exp(2tx-t^2)\exp(-tD)(2x-D)^k\cdot 1. \qquad \text{...(25)}$$

Since

$$[D^s, x]=\begin{cases}0, & \text{for } s=0,\\ sD^{s-1}, & \text{for } s\geqslant 1,\end{cases} \qquad \text{...(26)}$$

we have $\exp(-tD)(2x-D)=(2x-D)\exp(-tD)+[\exp(-tD), 2x-D]=\{2(x-t)-D\}\exp(-tD)$. Iteration of this result yields $\exp(-tD)(2x-D)^k.1=\{2(x-t)-D\}^k\exp(-tD)\cdot 1=\{2(x-t)-D\}^k\cdot 1=H_k(x-t)$, $k\geqslant 1$. However, this result is also true for $k=0$. Hence, when $k\geqslant 0$,

$$\exp(-tD)H_k(x)=\exp(-tD)(2x-D)^k\cdot 1=H_k(x-t), \qquad \text{...(27)}$$

and

$$S(k, x, t)=\exp(2xt-t^2)H_k(x-t). \qquad \text{...(28)}$$

As a simple corollary of (28), we can obtain the generating function [see (21)] for $H_n(x)$, by choosing $k=0$. The importance of (22) lies in the fact that it is the basis for deriving a bilinear generating fuction for the Hermite polynomials (see, *e.g.*, [5, *p*. 197]). Moreover, by applying (22) to any known generating relation, one can sometimes obtain a new result (see, *e.g.*, [5, p. 198]).

We next prove (23) using induction. Clearly, it is true for $n=0, 1, 2$. Hence let us assume that it is also true for some positive integer m. Using (3*a*), we have, $H_{m+1}(x)=(2x-D)H_m(x)=2^m(2x-D)\exp(-D^2/4)x^m$. Now $(2x-D)\exp(-D^2/4)=\exp(-D^2/4)(2x-D)+[2x-D, \exp(-D^2/4)]$. From (26), $[x, \exp(-D^2/4)]=\exp(-D^2/4)]\,D$. Moreover, $[D, \exp(-D^2/4)]=0$. Hence $(2x-D)\exp(-D^2/4)=\exp(-D^2/4)\,2x=2\exp(-D^2/4)\,x$. Thus $H_{m+1}(x)=2^{m+1}\exp(-D^2/4)x^{m+1}$. Therefore, (23) is true for $n=m+1$), if it is true for $n=m$. Hence, by induction, Bell's formula (23) is true for all $n\geqslant 0$.

It is clear from this section that (1) may not be that efficient in obtaining identities involving the Hermite polynomials, while it is elegant in developing certain properties of $H_n(x)$. Every starting point has its own limitations. For more identities involving the Hermite polynomils, see, for example, [1], [5], [7] and [9].

10. CONCLUSIONS

We have shown that all the standard properties of the Hermite polynomials can be derived from the simple, elegant and beutiful relation (1). It is rather strange that this wonderful relation has not found a place in the masterpieces from Szegö [9]. Rainville [5] and others (*e.g.*, [1], [3], [6]).

We sincerely feel that this relatian from Arfken [2] deserves the status of a new starting point for developing the properties of the Hermite polynomials. However, it may not be very powerful in deriving identities involving the Hermite polynomials.

Acknowledgments

It is a pleasure to thank Dr. S. D. Rindani for fruitful communication, Professor T. Nagarajan, Dr. M. Rajasekaran and Ms. G. Janhavi for useful discussions, the referee for making constructive suggestions, and the University Grants Commission, New Delhi, for support through its COSIST programme. This work has been classroom tested and we are grateful to our students for their enthusiastic response. With all humility, we dedicate this work to Professor George Arfken whose beutiful book [2] and its earlier edition (1970) have always been a source of inspiration to us.

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Jñānābha, Vol. 20, 1990

On Simultaneous Operational Calculus Involving a Product of Fox's H-Function and the Multivariable H-Function

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(Received : April 25, 1988, Revised : November 14, 1888)

ABSTRACT

The object of this paper is to obtain new operational relations between the original and the image for the two-dimensional Laplace transforms that involve Fox's H-function and the multivariable H-function. The result obtained here also provides a unification of the bivariate Laplace transforms for H-functions given in a recent paper (*cf.*[1]).

1. INTRODUCTION

The integral equation

$$\phi(p,q)=pq\int_0^\infty\int_0^\infty e^{-px-qx} f(x,y)\,dx\,dy,\ Re(p)>0,\ Re(q)>0. \qquad ...(1.1)$$

represents the classical Laplace transform in two variables and $\phi(p, q)$ and $f(x; y)$ related by (1.1) are said to be operationally realated to each other. $\phi(p, q)$ is called the image and $f(x, y)$ the original.

Symbolically, we can write

$$\phi(p,q) \doteqdot f(x,y) \text{ or } f(x,y) \doteqdot \phi(p,q) \qquad ...(1.2)$$

and the symbol $\doteqdot$ is called operational.

The multivariable H-function has been recently introduced by Srivastava and panda [5] and is defined and represented in the following manner (see also [6, *p.* 251]) :

$$H^{0,\,u\,:\,(M',N');\,\dots;\,(M^{(r)},N^{(r)})}_{v,\,w\,:\,(P',Q');\,\dots;\,(P^{(r)},Q^{(r)})}\left(\begin{array}{l}[(a):A',\dots,A^{(r)}]:\\ [(c):C',\dots,C^{(r)}]:\end{array}\begin{array}{l}[(b'):B'];\dots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\dots;[(d^{(r)}):D^{(r)}];\end{array}\ z_1,\dots,z_r\right)$$

$$=(2\pi\omega)^{-r}\int_{L_1}\cdots\int_{L_r} U_1(s_1)\,..U_r(s_r)V(s_1,\ \ldots,\ s_r)\ z_1^{\ s_1}\ldots z_r^{\ s_r}\cdot ds_1\ldots ds_r,\ \omega=\sqrt{-1} \qquad \ldots(1.3)$$

where

$$U_i(s_i)=\prod_{j=1}^{M^{(i)}}\Gamma(d_j^{(i)}-D_j^{(i)}\ s_i)\prod_{j=1}^{N^{(i)}}\Gamma(1-b_j^{(i)}+B_j^{(i)}s_i)\cdot\left\{\prod_{j=1+M^{(i)}}^{Q^{(i)}}\Gamma(1-d_j^{(i)}+D_j^{(i)}\ s_i)\prod_{j=1+N^{(i)}}^{P^{(i)}}\Gamma(b_j^{(i)}-B_j^{(i)}\ s_i)\right\}^{-1} \qquad \ldots(1.4)$$

$$V(s_1,\ \ldots,\ s_r)=\prod_{j=1}^{u}\Gamma(1-a_j+\sum_{j=1}^{r}A_j^{(i)}\ s_i)\cdot\left\{\prod_{j=u+1}^{v}\Gamma(a_j-\sum_{i=1}^{r}A_j^{(i)}\ s_i)\prod_{j=1}^{w}\Gamma(1-c_j+\sum_{i=1}^{r}C_j^{(i)}\ s_i)\right\}^{-1} \qquad \ldots(1,5)$$

an empty product is interpreted as unity, also u, v, w, $N^{(i)}$ $M^{(i)}$, $P^{(i)}$ and $Q^{(i)}$ are such that $v\geqslant u\geqslant 0$, $w\geqslant 0$, $Q^{(i)}\geqslant M^{(i)}\geqslant 0$, $P^{(i)}\geqslant N^{(i)}\geqslant 0$, and $A_j^{(i)}$, $j=1, \ldots, V$; $B_j^{(i)}, j=1, \ldots, P^{(i)}$; $C_j^{(i)}, j=1, \ldots, W$; $D_j^{(i)}, \ldots, Q^{(i)}$ are positive quantities.

The contours L_1 are defined suitably and the poles of the integrand are assumed to be simple.

The multiple integral in (1·3) converges absolutely

if $|\arg(z_i)| < T_i\ \pi/2,\ i=1, \ldots, r$

when

$$T_i=-\sum_{j=u+1}^{v}A_j^{(i)}+\sum_{j=1}^{N^{(i)}}B_j^{(i)}-\sum_{j=1+N^{(i)}}^{P^{(i)}}B_j^{(i)}-\sum_{j=1}^{w}C_j^{(i)}+\sum_{j=1}^{M^{(i)}}D_j^{(i)}-\sum_{j=1+M^{(i)}}^{Q^{(i)}}D_j^{(i)}>0,\ i=1,\ \ldots,\ r, \qquad \ldots(1.6)$$

(a) denotes the sequence of v parameters $a_1, \ldots, a_v$, for each $i=1,\ldots, r$; $(b^{(i)})$ abbreviates the sequence of $P^{(i)}$, with similar interpretations for (c), (d) $(d^{(i)})$, etc., $i=1,\ldots, r$; will be understood, for example, that $b^{(1)}=b'$, $b^{(2)}=b''$ and so on.

The following result is required in our investigation.

The series representation of Fox's H-function (see [3] and [4] see also [6], *p*. 12)

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$$H_{u,\,d}^{m,\,n}\left[z \,\middle|\, \begin{matrix}(e_u,\,E_u)\\ (f_d,\,F_d)\end{matrix}\right]=\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,F_G}\;\phi(g_s)\;z^{g_s} \qquad \ldots(1\cdot7)$$

where

$$\phi(g_s)=\prod_{j=1,\,j\neq G}^{m}\Gamma(f_j-F_j\,g_s)\prod_{j=1}^{n}\Gamma(1-e_j+E_j\,g_s)$$

$$\cdot\left\{\prod_{j=1+m}^{d}\Gamma(1-f_j+F_j\,g_s)\prod_{j=1}^{u}\Gamma(a_j-E_j\,g_s)\right\}^{-1} \qquad \ldots(1\cdot8)$$

and $\quad g_s=(f_G+s)/F_G \qquad \ldots(1\cdot9)$

For the sake of brevity,

$$T=\sum_{1}^{n}E_l-\sum_{n+1}^{u}E_l+\sum_{1}^{m}F_l-\sum_{m+1}^{d}F_l \qquad \ldots(1\cdot10)$$

$$\left.\begin{aligned}&\theta=f_l/F_l,\ l=1,\ldots,m\\ &\phi=(e_{l'}-1)/E_{l'},\ l'=1,\ldots,n\end{aligned}\right\} \qquad \ldots(1\cdot11)$$

$$\left.\begin{aligned}&\theta_i=d_j^{(i)}\,/\,D_j^{(i)},\ j=1,\ldots,\ M^{(i)}\\ &\phi_i=(1-b_j^{(i)})\,/\,B_j^{(i)},\ j=1,\ldots,N^{(i)},\ i=1,\ldots,r\end{aligned}\right\}. \qquad \ldots(1\cdot12)$$

Also we use the notation

$$H_{v,\,w\,:\,(P',\,Q');\,\ldots,\,(P^{(r)},\,Q^{(r)})}^{0,\,u\,:\,(M',\,N');\,\ldots,\,(M^{(r)},\,N^{(r)})}\left(z_1,\ldots,z_r\right) \qquad \ldots(1\cdot13)$$

to denote the first member of (1·1).

In this paper we shall obtain correspondances, involving a product of Fox's H-function and the multivariable H-function, between the original and the image in two variables.

In what follows we shall denote the original variables by x and y and the transformed variables by p and q. The notations employed are those Ditkin and Prudnikov's operational calculus.

Theorem 1

With T_i, T, θ, ϕ, θ_i and ϕ_i given by (1·6), (1·10), (1·11) and (1·12) respectively, let $T_i>0$, $T>0$, $|\arg(z_i)|<T_i\,\pi/2$, $|\arg(z)|<T\pi/2$, $k>0$, $h_i>0$, $i=1,\ldots,r$, and

(i) $\operatorname{Re}\left(\sigma+k\theta+\sum_{i=1}^{r}h_i\,\theta_i\right)>0$, $\operatorname{Re}(\rho)>0$,

(ii) $\operatorname{Re}\left(\rho-\sigma-k\phi-\sum_{i=1}^{r}h_i\,\phi_i\right)<3/4$.

Also let $0 \leqslant n \leqslant u$, $0 \leqslant m \leqslant d$, and

(*iii*) $\mathrm{Re}(p) > 0$.

$$p^{-1/2}(pq)^{\sigma/2-\rho+1} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,F_G}\, \phi(g_s)\, z^{g_s}\, (pq)^{kg_s/2}$$

$$\cdot H^{0,0\,:\,(M',N');\ldots\ldots;(M^{(r)},N^{(r)})}_{v,w\,:\,(P',Q');\ldots\ldots;(P^{(r)},Q^{(r)})} \left(z_1(\sqrt{pq})^{h_1}, \ldots, z_r(\sqrt{pq})^{h_r} \right)$$

$$\mathrel{\substack{\cdot\cdot\\=\\\cdot\cdot}} \frac{(4\times y)^{\rho-\sigma/2-1/2}}{\sqrt{\pi y}} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)_s}{s!\,F_G}\, \phi(g_s)\, z^{g_s}\, (4xy)^{-kg_s/2}$$

$$\cdot H^{0,0\,:\,(M',N');\ldots\ldots;(M^{(r)},N^{(r)})}_{v+1,w\,:\,(P',Q');\ldots\ldots;(P^{(r)},Q^{(r)})} \left(\begin{matrix} [(a : A',\ldots\ldots,A^{(r)}],[2\rho-\sigma-kg_s : h_1,\ldots,h_r) : \\ [(c) : C',\ldots\ldots,C^{(r)}] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : B'];\ldots\ldots;[(b^{(r)}) : B^{(r)}]; \\ [(d') : D'];\ldots-;[(d^{(r)}) : D^{(r)}]: \end{matrix} \cdot z_1(2\sqrt{xy})^{-h_1},\ldots\ldots,z_r(2\sqrt{xy})^{-h_r} \right) \qquad \ldots(2.1)$$

Proof. The Laplace transform of a product of Fox's H-function and multivariable H-function is given by

$$\int_0^\infty e^{-pt}\, t^{\sigma-1}\, H^{m,n}_{u,d}\left[zt^k \middle| \begin{matrix} (e_u, E_u) \\ (f_d, F_d) \end{matrix} \right]$$

$$\cdot H^{0,0\,:\,(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w\,:\,(P',Q');\ldots;(P^{(r)},Q^{(r)})} \left(\begin{matrix} [(a) : A',\ldots,A^{(r)}] \\ [(c) : C',\ldots,C^{(r)}],\ [1-\sigma-kg_s : h_1,\ldots,h_r] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : B'];\ldots;[(b^{(r)}) : B^{(r)}]; \\ [(d') : D'];\ldots;[(d^{(r)}) : D^{(r)}); \end{matrix}\ z_1 t^{h_1},\ldots,z_r t^{h_r} \right) dt$$

$$= p^{-\sigma} \sum_{G=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,F_G}\, \phi(g_s)\, z^{g_s}\, p^{-kg_s}$$

$$\cdot H^{0,0\,:\,(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w\,:\,(P',Q');\ldots;(P^{(r)},Q^{(r)})} \left(z_1 p^{-h_1},\ldots;z_r p^{-h_r} \right) \qquad \ldots(2.2)$$

where $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\sigma + k\theta + \sum_{i=1}^{r} h_i\theta_i) > 0$, $\mathrm{Re}(\sigma + k\phi + \sum_{i=1}^{r} h_i\phi_i) < 0$, $k > 0$,

$h_i > 0$, $|\arg(z)| < T\pi/2$, $|\arg(z_i)| < T_i\pi/2$, $T > 0$, $T_i > 0$.

The result in (2.2) can be established by substituting the series (1.7) for Fox's H-function and changing the order of integration and summation (which is justified due to the absolute convergence of the integral involved in the process under the conditions mentioned), then evaluating the inner integral and using the definitions (1.1), we arrive at the result given in (2.1).

On writting $(pq)^{-1/2}$ for p, multiplying both sides of (2.2) by $p^{-1/2}$ $(pq)^{1-\rho}$ and then interpreting it with the help of a known result ([2], *p.*144 eqn. (3.26)), we get

$$\frac{(4xy)^{\rho/2-1/4}}{\sqrt{\pi y}}\int_0^\infty t^{\sigma-\rho+1/2}\, J_{2\rho-1}[(64xyt^2)^{1/4}]\, H^{m,n}_{u,d}\left[zt^k\middle|\begin{matrix}(e_u, E_u)\\(f_d, F_d)\end{matrix}\right]$$

$$\cdot H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w+1:(P,Q');\ldots;(P^{(r)},Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}]: \\ [(c):C',\ldots,C^{(r)}]:[1-\sigma-kg_s:h_1\ldots,h_r]\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}]; \\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}]]\end{matrix}\; z_1t^{h_1},\ldots,z_r\,t^{h_r}\right)dt$$

$$\doteqdot p^{-1/2}(pq)^{-\sigma/2-\rho+1}\sum_{G=1}^{m}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!F_G}\,\phi(g_s)\,z^{g_s}\,(pq)^{kg_s/2}$$

$$\cdot H^{0,0:(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,w:(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(z_1(\sqrt{pq})^{h_1},\ldots,z_r(\sqrt{pq})^{h_r}\right), \qquad \ldots(2.3)$$

provided $\mathrm{Re}(\rho)>0$,

Now evaluating the left hand side integral of (2.3) by the process mentioned in (2.2) to obtain the desired result. Hence (2.1) is proved.

3. SPECIAL CASES

Letting $k\to 0$, from (2.1) we obtain (after a little simplification) the following bivariate Laplace transform for the multivariable *H*-function in the elegant form.

Theorem 2

With T_i, θ_i and ϕ_i given by (1.6) and (1.12) respectively, let $T_i>0$, $|\arg(z_i)|<T_i\,\pi/2$, $h_i>0$, $i=1,\ldots,r$ and

(*i*) $\quad \mathrm{Re}(\sigma+\sum_{i=1}^{r} h_i\theta_i)>0,$

(*ii*) $\quad \mathrm{Re}(\rho-\sigma-\sum_{i=1}^{r} h_i\phi_i)<3/4.$

Also let (*iii*) $\quad \mathrm{Re}(\rho)>0$, and $\mathrm{Re}(p)>0$.

Then

$$p^{-1/2}(pq)^{\sigma/2-\rho+1}\, H^{0,\,0\,:\,(M',N');\ldots;(M^{(r)},N^{(r)})}_{v,\,w\,:\,(P',Q');\ldots;(P^{(r)},Q^{(r)})}\left(z_1(\sqrt{pq})^{h_1},\ldots,z_r(\sqrt{pq})^{h_r}\right)$$

$$\doteqdot (4xy)^{\rho-\sigma/2-1/2}\, H^{0,\,0\,:\,(M',N');\ldots;M^{(r)},N^{(r)})}_{v+1,\,w\,:(P',Q');\ldots;P^{(r)}\,Q^{(r)})}\left(\begin{matrix}[(a):A',\ldots,A^{(r)}],\\ [(c):C'\ldots,C^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[2\rho-\sigma:h_1,\ldots,h_r].\,[(b'):B'];\ldots;[(b^{(r)}):B^{(r)}];\\ [(d'):D'];\ldots;[(d^{(r)}):D^{(r)}];\end{matrix}\; z_1(2\sqrt{xy})^{-h_1},\ldots,z_r(2\sqrt{xy})^{-h_r}\right) \quad \ldots(3.1)$$

The special cases of the bivariate Laplace transforms (3.1), when $r=1$, $r=2$, were given by Chaurasia ([1] *p*.86, eqn.(2.1) and *p*.89.eqn. (3.2)).

The bivariate Laplace transforms for a product of Fox's H-function and the multivariable *H*-function given by Theorem 1 posseses manifold generality. On specializing the parameters the multivariable *H*-function may be transformed into the *G*-functions, *E*-functions, *H*-functions, hypergeometric functions, Legendre functions, Kampe' de Fe'riet functions, Lauricella's functions, Appell's functions, Bessel functions, and several other higher transcedental functions in one, two or more arguments. Therefore, the results established in this paper are of general nature and hence encompass several cases of interests.

ACKNOWLEDGEMENTS

I am thankful to Professor H. M. Srivastava for his keen interest in the preparation of this paper and to the University Grants Commission, New Delhi, India, for financial support.

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Jñānabha, Vol, 20, 1990

A Result on the Derivative of Meyer Konig and Zeller Operators

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(Received : May 5, 1988, Revised : October 4, 1989)

ABSTRACT

The purpose of the present note is to give an estimate on the derivative of Meyer Konig and Zeller operators, which is sharper than those of Muller [2] and Singh [4].

1. Introduction

In M. Muller [2], the derivative of Meyer Konig and Zeller positive linear operator (Mn) is defined as

$$(1{\cdot}1) \qquad (Mn'f)(x)=(Mnf)'(x)$$

$$=\sum_{k=0}^{\infty} m_{n-1,\,k}(x)\, f'(\theta_{nk}),\ (k/n+k<\theta_{nk}<\ldots$$

$$\ldots<k+1/n+k+1),\ \text{all } x\in[0,1)$$

$$(1.2) \qquad (Mn'f)(1)=(\mathrm{Mnf})'(1)=f'(1),\ (n\geqslant 2).$$

The following estimate on the order of convergence of the derivative of a function f by $Mn'f$ was given by Muller.

Theorem 1 (Müller [2]). Let $f\in C^2\,[0,1]$ and $w(f^H;\delta)$ is the modulus of continuity of f^H. Then for $n\geqslant 3$.

$$(1{\cdot}3) \qquad \| Mn'f-F' \|_{\infty} \leqslant \mu_n\, w(f^H;\, 1/\sqrt{n-2}) + |f^H|/n,$$

where

$$(1{\cdot}4) \qquad \mu_n=\left(1+\frac{4\sqrt{3}}{9}\right)\frac{1}{n}+\left(\frac{3\sqrt{3}}{9}+\frac{4}{27}\right)\frac{1}{\sqrt{n-2}}+\frac{\sqrt{n-2}}{n^2}.$$

The object of the present paper is to improve the estimate (1·4).

We prove the following theorem.

Theorem 2. Let $f\in C^2\,[0,1]$ and $w(f^H;\delta)$ be the modulus of continuity of f^H. Then for $n\geqslant 2$.

$$(1\cdot5) \qquad \| M_n' f - f' \|_\infty \leqslant d_n\, w(f;\, H_1/\sqrt{n-1}) + \| f^H \|_\infty / n,$$

where

$$d_n = \begin{cases} 0\cdot89 & \text{if } \quad n=2 \\ \left(\frac{1}{2}+\frac{2\sqrt{3}}{9}\right)\frac{1}{n}+\left(\frac{\sqrt{3}}{9}+\frac{1}{8}+\frac{2}{27}\right)\frac{1}{\sqrt{n-1}} & \\ \qquad +\frac{\sqrt{n-1}}{2\,n^2}-\lambda(n) & \text{other wise ,} \end{cases}$$

where

$$\lambda(n) = \frac{n^2-2n-4}{54(\sqrt{n-1})(n^2-2n)^2}$$

The above estimate (1·2) is also sharper then the result ([4], theorm 2).

2. Preliminary Result

We use the following results to prove the theorem.

Proposition 1. (Anastassiou, [1]). For all $t, x_0 \in [a, b]$, one gets.

$$(2\cdot1) \qquad \int_{x_0}^{t} [\,|\, y-x_0 \,|\, /\delta]\; dy \leqslant \left(\frac{t-x_0}{2\delta}\right)^2 + \left|\frac{t-x_0}{2}\right| + \frac{\delta}{8}\,.$$

where $[\cdot]$ indicates the ceilling of the number.

Proposition 2 (Sikkemma [3]). On $[0, 1]$, one has for $n \geqslant 2$.

$$(2\cdot2) \qquad (M_{n-1}\, 1)\,(x) = 1, \;\; (M_{n-1}\, t)\,(x) = x,$$

$$(2\cdot3) \qquad (M_{n-1}\,(t-x)^2)\,(x) \leqslant x\,\frac{(1-x)^2}{n} + \frac{x^2(1-x)\,(2-x)}{n^2}$$

$$= d_1\,(n, x) \quad \text{(say)},$$

where

$$(2.4) \qquad d_1(n, x) = \begin{cases} \leqslant 0\cdot1113, & \text{if } \; n=2. \\ \leqslant \frac{2}{27(n-1)} - \frac{2\lambda(n)}{\sqrt{n-1}}\,, & n \geqslant 3. \end{cases}$$

3. Proof of Theorem 2

We know that

$$f'(\theta_{nk}) - f'(x) = (\theta_{nk}-x)\; f^H(x)$$
$$+ \int_x^{\theta_{nk}} \{ f^H(\eta) - f^H(x) \} d\eta.$$

Using (1·1), we get for all $x \in [0, 1]$,

$$|\,(M'_n f)\,(x) - f'(x)\,| = \Big|\, \sum_{k=0}^{\infty} m_{n-1,\,k}\,(x)\; [\, f'(\theta_{nk}) - f'(x)]\, \Big|\,,$$

$$\leqslant | f^H(x) | \cdot | \sum_{k=0}^{\infty} m_{n-1,k}(x)(\theta_{nk}-x) | + \sum_{k=0}^{\infty} m_{n-1,k}(x) | \left| \int_x^{\theta_{nk}} \{(f^H(\eta)) - f^H(x)\} \, d\eta \right|.$$

From the inequalities

$$\left| \theta_{nk} - \frac{k}{n+k-1} \right| \leqslant \frac{1}{n},$$

$$\left| f^H(\eta) - f^H(x) \right| \leqslant \left[\frac{|\eta - x|}{\delta} \right] w(f^H; \delta)$$

and (2·1), we get

$$| M'_n f)(x) - f'(x) |$$

$$\leqslant \| f^H \|_\infty \cdot \left| \sum_{k=0}^{\infty} m_{n-1,k}(x) \left(\theta_{nk} - \frac{k}{n+k-1} + \frac{k}{n+k-1} - x\right) \right|$$

$$+ \ldots + w(f^H; \delta) \sum_{k=0}^{\infty} m_{n-1,k}(x) \left(\frac{|\theta_{nk} - x|}{2} + \frac{(\theta_{nk}-x)^2}{2\delta} + \frac{\delta}{8} \right),$$

$$= S_1 + S_2, \text{ (say)}$$

Obviously

$$S_1 = \| f^H \|_\infty / n$$

and
$$S_2 = w(f^H; \delta) \sum_{k=0}^{\infty} m_{n-1,k}(x) \left(\frac{|\theta_{nk} - x|}{2} + \frac{(\theta_{nk}-x)^2}{2\delta} + \frac{\delta}{8} \right),$$

$$\leqslant w(f^H, \delta) \sum_{k=0}^{\infty} m_{n-1,k}(x) \left[\frac{1}{2n} + \frac{1}{2} \left| \frac{k}{n+k-1} - x \right| + \frac{\delta}{8} + \ldots + \frac{1}{2\delta} \left(\frac{1}{n} + \left| \frac{k}{n+k-1} - x \right| \right)^2 \right];$$

$$= w(f^H; \delta) \sum_{k=0}^{\infty} m_{n-1,k}(x) \left[\left(\frac{1}{2n} + \frac{1}{2n^2\delta} + \frac{\delta}{8} \right) \cdot + \left(\frac{1}{n\delta} + \frac{1}{2} \right) \left(\left| \frac{k}{n+k-1} - x \right| \right) + \frac{1}{2\delta} \left(\frac{k}{n+k-1} - x \right)^2 \right]$$

Choosing $\delta = 1/\sqrt{n-1}$, $(n \geqslant 2)$ and using (2·2) and (2·3), we get

$$S_2 \leqslant w\left(f''; 1/\sqrt{n-1}\right) \left[\left(\frac{1}{2n}+\frac{\sqrt{n-1}}{2n^2}+\frac{1}{8\sqrt{n-1}}\right) + \left(\frac{\sqrt{n-1}}{n}+\frac{1}{2}\right) \cdots \sqrt{d_1(n, x)}+\frac{\sqrt{n-1}}{2}\, d_1(n, x)\right],$$

and consequently the estimate (1·5) is valid for all $x \in [0, 1]$ and $n \geqslant 2$.

Furthermore, at $x=1$, the estimate (1·5) holds true because of (1·2).

This completes the proof of Theorem 2.

ACKNOWLEDGEMENTS

The authors are thankful to Prof. O. P. Vershney, University of Roorkee, for his valuable help and suggestion.

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Jñānabha, Vol, 20, 1990

Linear Flow of Heat in an Anisotropic Finite Solid Moving in a Conducting Medium and the Multivariable H-Function

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ABSTRACT

This paper employs the H-function, the multivariable H-function and the generalized prolate spheroidal wave function in obtaining a formal solution of the partial differential equation related to a problem of linear flow of heat in an anisotropic finite solid moving in a conducting medium. This type of problems occour mainly in soil mechanics, wood technology and the mechanics of solids of fibrous structure. The results of this paper unify and extend a large number of results established by various earlier workers.

1. INTRODUCTION

As an example of the application of certain product of the H-function the multivariable H-function and the generalized prolate spheroidal wave function in applied mathematics, we shall consider a problem of linear heat flow in an anisotropic finite solid moving in a conducting medium, given by the following partial differential equation (see[5])

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right]-\frac{p'\,c'\,\nu}{\lambda}\;\frac{\partial U}{\partial x}+\frac{Q(x)}{\lambda}=\frac{p'\,c'}{\lambda}\;\frac{\partial U}{\partial t} \qquad \ldots(1.1)$$

with the law of conductivity $K=\lambda\,(1-x^2)$, $Q(x)$ is the intensity of a continuous source of heat situated inside this solid. Let the initial temperature of the rod be given by

$$U(x,\,0)=F(x) \qquad \ldots(1.2)$$

Letting

$$\nu=(\alpha-\beta)/q',\; Q(x)=-(\alpha+\beta)\lambda x\frac{\partial U}{\partial x}-(s'x)^2\lambda U,\; q'=p'c'/\lambda,$$

a solution of (1.1) takes the form

$$U(x, t)=\sum_{l=0}^{\infty} T_l \exp(-G_l t)\, \phi_l^{\alpha,\beta}(s', x) \qquad \ldots(1.3)$$

where $\qquad G_l=(l+p')\,(l+p'+\alpha+\beta+1/q'.$

Here we shall consider

$$F(x)=(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}\; H_{p,q}^{m,n}\left[z(1-x)^h\,(1+x)^k \,\middle|\, \begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right]$$

$$\cdot H_{A,\,C\,:\,(P',\,Q');\,\ldots;\,(P^{(r)}\,Q^{(r)})}^{0,\,B\,:\,(M',\,N');\,\ldots;\,(M^{(r)},\,N^{(r)})}\left(z_1(1+x)^{h_1}(1+x)^{k_1}, \ldots, z_r(1-x)^{h_r}(1+x)^{k_r}\right) \qquad \ldots(1.4)$$

where the multivariable H-function of several complex variables introduced by Srivastava and Panda ([7] and [8]; see also [9], *p*. 251).

The solution of the following differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+[(\beta-\alpha)-(\alpha+\beta+2)x]\,\frac{dy}{dx}+[\xi(s')+(s'x)^2]y=0 \qquad \ldots(1.5)$$

is the generalized prolate spheroidal wave function (see, *e.g.*, [3], *p*. 107, eqn.(2.11)) which has been denoted as

$$\phi_n^{\alpha,\beta}(s', x)=\sum_{p=0}^{\infty} R_{p,n}^{\alpha,\beta}(s')\, P_{p+n}(x) \qquad \ldots(1.6)$$

where $\qquad s'=0$ and $\xi(0)=(n+p)\,(\alpha+\beta+n+p+1),\ p\geqslant 0.$

The following results are required in our investigation :

The series representation of Fox's H-function (see [2] and [6]; see also [9, *p*. 12]) :

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(e_p, E_p)\\(f_q, F_q)\end{matrix}\right]=\sum_{g=1}^{m}\ \sum_{s=0}^{\infty}\frac{(-1)^s}{s!F_g}\,\phi(\eta_s)\, z^{\eta_s} \qquad \ldots(1.7)$$

where

$$\phi(\eta_s)=\prod_{j=1,\,j\neq g}^{m}\Gamma(f_j-F_j\,\eta_s)\prod_{j=1}^{n}\Gamma(1-e_j+E_j\,\eta_s)$$

$$\cdot\left\{\prod_{j=m+1}^{q}\Gamma(1-f_j+F_j\,\eta_s)\prod_{j=n+1}^{p}\Gamma(e_j-E_j\,\eta_s)\right\}^{-1}$$

and $\qquad \eta_s=(e_g+s)/F_g \qquad \ldots(1.8)$

The orthogonality property of the generalized prolate spheroidal wave fucntion ([3], *p*. 107, eqn. (3.1)) :

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \phi_{v}^{\alpha,\beta} (s', x) \phi_{w}^{\alpha,\beta} (s', x)\, dx = N_{v,w}^{\alpha,\beta} \delta_{v,w} \quad \ldots(1.8)$$

where

$$N_{v,w}^{\alpha,\beta} = \sum_{p=0}^{\infty} (R_{p,v}^{\alpha,\beta}(s'))^2 \frac{2^{\alpha+\beta+1} \Gamma(v+\alpha+p+1) \Gamma(v+\beta+p+1)}{(2v+2p+\alpha+\beta+1) \Gamma(v+p+1) \Gamma(v+p+\alpha+\beta+1)}$$

and $\delta_{v,w}$ is the Kronecker delta.

2. THE MAIN INTEGRAL FORMULA

The integral to be evaluated here is

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \phi_{w}^{\alpha,\beta} (s', x) H_{p,q}^{m,n} \left[z(1-x)^{h} (1+x)^{k} \left| \begin{matrix} (e_p, E_p) \\ (f_q, F_q) \end{matrix} \right. \right]$$

$$\cdot H_{A,\,C\,:\,(P',\,Q');\,\ldots;\,P^{(r)},\,Q^{(r)}}^{0,\,B\,:\,(M'\,N');\,\ldots;\,(M^{(r)},\,N^{(r)})} \left(z_1 (1-x)^{h_1} (1+x)^{k_1}, \ldots, z_r (1-x)^{h_r} (1+x)^{k_r} \right) dx$$

$$= \sum_{g=1}^{m} \sum_{s,\,v=0}^{\infty} \frac{(-1)^s}{s!\,F_g} \phi(\eta_s) z^{\eta_s} R_{v,w}^{\alpha,\beta}(s', x)\, 2^{1+\rho+\sigma+(h+k)\eta_s}$$

$$\cdot \sum_{u=0}^{v+w} (-v-w : u)(\alpha+\beta+v+w+1 : u) \{(\alpha+1 : u)\, u!\}^{-1}$$

$$\cdot H_{A+2,\,C+1\,:\,(P',\,Q');\,\ldots,\,(P^{(r)},\,Q^{(r)})}^{0,\,B+2\,:\,(M',\,N');\,\ldots,\,(M^{(r)},\,N^{(r)})} \left(\begin{matrix} [-u-\sigma-h\,\eta_s : h_1, \ldots, h_r],\ [-\rho-k : \eta_s : k_1 \ldots, k_r] \\ [(c) : \psi', \ldots, \psi^{(r)}] \end{matrix} \right.$$

$$[(a) : \theta' \ldots, \theta^{(r)}] : \qquad [(b') : \phi'];\, \ldots;\, [(b^{(r)}) : \phi^{(r)}];$$

$$[-1-\rho-\sigma-(h+k)\,\eta_s : h_1+k_1, \ldots, h_r+k_r] : [(d') : \delta'];\, \ldots;\, [(d^{(r)}) : \delta^{(r)}];$$

$$\left. z_1 2^{h_1+k_1}, \ldots, z_r 2^{h_r+k_r}, \right) \quad \ldots(2.1)$$

where
$$\mathrm{Re}\,(\rho + h f_l/F_l + \sum_{t=1}^{r} h_t d_j^{(t)}/\delta_j^{(t)}) > -1,$$

$\mathrm{Re}\,(\sigma + k f_l/F_l + \sum_{i=1}^{r} k_i d_j^{(i)}/\delta_j^{(i)}) > -1$, $\mathrm{Re}\,(\alpha) > -1$, $\mathrm{Re}\,(\beta) > -1$, $h_i > 0$, $k_i > 0$, $l = 1, \ldots, m$; $i = 1, \ldots, r$; $j = 1, \ldots, M^{(i)}$, $T_i > 0$, $|\arg(z_i)| < T_i \pi/2$

$$|\arg(z)| < T\pi/2 \ \left(T = \sum_{1}^{n} E_t - \sum_{n+1}^{p} E_t + \sum_{1}^{m} F_t - \sum_{m+1}^{q} F_t\right).$$

Evaluation of (2.1). The integral (2.1) can be evaluated in a routine manner with the help of a result of Chaurasia and Gupta ([1] *p*.82, eqn. (13)).

3. SOLUTION OF THE PROBLEM

The solution of the problem to be established here is

$$U(x,t) = \sum_{g=1}^{m} \sum_{v,\, s=0}^{\infty} \frac{(-1)^s}{s!\, F_g} \phi(\eta_s)\, z^{\eta s}\, R_{v,\ l}^{\alpha,\ \beta}(s', x)\, 2^{1+\rho+\sigma+(h+k)\eta s}$$

$$\cdot \sum_{l,\, u=0}^{v} \frac{(-v : u)(\alpha+\beta+v+1 : u)}{(\alpha+1 : u)\, u!} \left\{ N_{l,\ l}^{\alpha,\ \beta} \right\}^{-1} \phi_{l}^{\alpha,\ \beta}(s', x)\, e^{-v(v+\alpha+\beta+1)t/q'}$$

$$\cdot H_{A+2,\ C+1\ :\ (P',\ Q');\ \ldots;\ (P^{(r)},\ Q^{(r)})}^{0,\ B+2\ :\ (M',\ N');\ \ldots;\ (M^{(r)},\ N^{(r)})} \left(\begin{array}{l} [-u-\sigma-h\,\eta s : h_1, \ldots, h_r], [-\sigma-k\eta s : k_1, \ldots k] \\ [(c) : \psi' \ldots, \psi^{(r)}] \end{array} \right.$$

$$\begin{array}{ll} [(a) : \theta' \ldots, \theta^{(r)}] : & [(b') : \phi']; \ldots; (b^{(r)}) : \phi^{(r)}]; \\ [-1-\rho-\sigma-(h+k)\eta s : h_1+k_1, \ldots, h_r+k_r] : & [(d') : \delta']; \ldots; [(d^{(r)}) : \delta^{(r)}]; \end{array}$$

$$\left. z_1\, 2^{h_1+k_1}, \ldots, z_r\, 2^{h_r+k_r} \right), \quad \ldots(3.1)$$

valid under the conditions obtainable from (2.1).

Derivation of (3.1). If $t = 0$, then by virtue of (1.3) we have

$$F(x) = \sum_{l=0}^{\infty} T_l\, \phi_l^{\alpha,\ \beta}(s', x) \qquad \ldots(3\ 2)$$

($F(x)$ is given in (1.4)).

Equation (3.2) is valid since $F(x)$ is continuous and of bounded variation in the open interval $(-1, 1)$.

Multiplying both sides of (3.2) by $(1-x)^\alpha\ (1+x)^\beta\ \phi_w^{\alpha,\ \beta}(s', x)$, $\alpha > -1$ $\beta > -1$ and integrating from -1 to 1 with respect to x, and using the orthogonality property (1.9), the integral (2.1) and a lemma ([4], *p*. 57 eqn. (2)), after a little simplification we arrive at the required result.

4. SPECIAL CASES

On specializing the parameters, Fox's H-function and the multivariable H-function may be reduced to G-functions, E-functions, Lauricella's functions Legendre functions, Bessel functions hypergeometric functions, Appell's-functions, Kampe de Feriet's functions, and several other higher transcendental functions in one or more arguments. Therefore, $F(x)$ in (1.4) is of general character and may prove to be useful in several interesting situations appearing in the literature on applied mathematics and mathematical physics.

ACKNOWLEDGEMENTS

The author is thankful to Professor H. M. Srivastava for his help in the preparation of this paper, and to the University Grants Commission, New Delhi, for providing financial assistance for the work.

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Jñānabha; Vol. 20, 1990

Fractional Integral Operators Involving the Multivariable H-Function

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(Received : June 1, 1988, Revised : November 18, 1988)

ABSTRACT

In this paper we introduce two univariable and two multivariable fractional integral oferators involving the multivariable *H*-function. The fractional integral operators of another multivariable *H*-function are then obtained in the form of four theorems.

1. INTRODUCTION

The *H*-function of several complex variables $z_1, \ldots, z_r$ is defined by Srivastava and Panda ([2], [3]) by means of the following multiple contour integrals [1, *p*. 251, eq. (*C*.1)] :

$$(1{\cdot}1)\quad H\,{}^{0,\,\varepsilon\;:\;(u',\,v');\;\ldots;\;(u^{(r)},\,v^{(r)})}_{A,\,C\;:\;(B',\,D'];\;\ldots;.\;(B^{(r)},\,D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: & [(b'),\phi');\ldots;[(b^{(r)}),\phi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}]: & [(d'),\delta'];\ldots;[(d^{(r)}),\delta^{(r)}]\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} W(s_1,\ldots,s_r)\prod_{i=1}^{r}\{T_i(s_i)z_1^{S_1}\,ds_i\},\omega=\sqrt{-1.}$$

For the definitions of the functions *W* and *T* and also the conditions of existence of the multivariable *H*-function, we refer to [1, *p*. 251-253, eqns. $(C{\cdot}2)-(C{\cdot}8)$]. The various conditions of existence of the multivariable *H*-function are assumed to be satisfied for the multivariable *H*-functions occurring in this paper.

2. GENERALIZED UNIVARIABLE FRACTIONAL INTEGRAL OPERATORS

Definitions. Let

$$(2\cdot1)\qquad R[\,f(x)]=R\left[\begin{matrix}\eta,\ \alpha;\ k\\ a_1,\ldots,\ a_r;\ x\end{matrix}\ ;f(x)\right]$$

$$=k\,x^{-\eta-k\alpha-1}\int_0^x t^{\eta}(x^k-t^k)^{\alpha}\ H[a_1\,V,\ \ldots,\ a_r\,V]\,f(t)\,dt$$

and

$$(2\cdot2)\qquad K[\,f(x)]=K\left[\begin{matrix}\eta,\ \alpha;\ \mu\\ a_1,\ \ldots,\ a_r;\ x\end{matrix}\ ;f(x)\right]$$

$$=\mu\ x^{\eta}\int_x^{\infty} t^{-\eta-\mu\alpha-1}\,(t^{\mu}-x^{\mu})^{\alpha}\ H_1[a_1V',\ldots,\ a_rV']\,f(t)\,dt$$

where

$$(2\cdot3)\qquad V=\left(\frac{t^k}{x^k}\right)^{\rho}\left(1-\frac{t^k}{x^k}\right)^{\sigma},$$

$$(2\cdot4)\qquad V'=\left(\frac{x^{\mu}}{t^{\mu}}\right)^{\rho'}\left(1-\frac{x^{\mu}}{t^{\mu}}\right)^{\sigma'},$$

$$(2\cdot5)\qquad H[x_1,\ \ldots,\ xr]=H^{0,\ \varepsilon_1\ :\ (m',\ n')\ ;\ \ldots;\ (m^{(r)}\ n^{(r)})}_{A_1,\ C_1\ :\ [M',\ N'];\ \ldots;\ [(M^{(r)},\ N^{(r)}]}\left(\begin{matrix}[(u)\ :\ \lambda',\ \ldots,\ \lambda^{(r)}]\ :\ [(e'),\ E'];\ ..;\ [(e^{(r)}),\ E^{(r)}];\\ [(v)\ :\ \gamma',\ \ldots,\ \gamma^{(r)}]\ :\ [(f'),\ F'];\ ..;\ [(f^{(r)},\ F^{(r)}]:\end{matrix}\ x_1,\ \ldots,\ xr\right),$$

and the H_1-function is obtained from (2·5) by taking $\varepsilon_1=0$ in it. The fractional integral operators (2.1) and (2.2) are valid under the following set of conditions :

(*i*) $1\leqslant p,\ q<\infty,\ \frac{1}{p}+\frac{1}{q}=1;$

(*ii*) $\mathrm{Re}(\eta)+k\rho\sum_{i=1}^{r}\tau^{(i)}>\max\left[-p^{-1},\ -q^{-1}\right]$

(*iii*) $\mathrm{Re}(\alpha)+\sigma\sum_{i=1}^{r}\tau^{(i)}-\frac{1}{q^k}$

(*iv*) $\mathrm{Re}(\alpha)+\sigma'\sum_{i=1}^{r}\tau^{(i)}>-\frac{1}{q^{\mu}}$

(*v*) $\mathrm{Re}(\eta)+\mu\rho'\sum_{i=1}^{r}\tau^{(i)}>\max\left(-p^{-1},\ -q^{-1}\right)$

(*vi*) $f(x)\epsilon L_p\,(0,\ \infty);$

where $\tau^{(i)}=\min_{1\leqslant j\leqslant m^{(r)}}\left[\mathrm{Re}(f_j^{(i)}/F_j^{(i)})\right].$

3. GENERALIZED MULTIVARIABLE FRACTIONAL INTEGRAL OPERATORS

Definition. Let

$$(3\cdot1)\qquad R^*[f(x_i)] \cdot R^*\left[\begin{matrix} \eta_i,\ a_i;\ k_i \\ a_i;\ x_i \end{matrix}\ ;\ f(x_i, \ldots, x_r)\right]$$

$$= \prod_{i=1}^{r} [k_i\, x_i^{-\eta_i - k_i a_i - 1}] \int_0^{x_1} \cdots \int_0^{x_r} \prod_{i=1}^{r} [t_i^{\eta_i} (x_i^{k_i} - t_i^{k_i})^{a_i}]$$

$$+ H[a_1 V_1, \ldots, a_r V_r]\, f(t_1, \ldots, t_r] \prod_{i=1}^{r} (dt_i),$$

and

$$(3\cdot2)\qquad K^*[f(x_i)] = K^*\left[\begin{matrix} \eta_i,\ a_i;\ \mu_i \\ a_i;\ x_i \end{matrix}\ ;\ f(x_1, \ldots, x_r)\right]$$

$$= \prod_{i=1}^{r} [\mu_i x_i^{\eta_i}] \int_{x_1}^{\infty} \cdots \int_{x_r}^{\infty} \prod_{i=1}^{r} [t_i^{-\eta_i - \mu_i a_i - 1} (t_i^{\mu_i} - x_i^{\mu_i})^{a_i}$$

$$\times H_1[a_1 V'_1, \ldots, a_r V'_r]\, f(t_1, \ldots, t_r) \prod_{i=1}^{r} (dt_i),$$

where

$$(3\cdot3)\qquad V_i = \left(\frac{t_i^{k_i}}{x_i^{k_i}}\right)^{\rho_i} \left(1 - \frac{t_i^{k_i}}{x_i^{k_i}}\right)^{\sigma_i},\ i = 1, \ldots, r;$$

$$(3\cdot4)\qquad V'_i = \left(\frac{x_i^{\mu_i}}{t_i^{\mu_i}}\right)^{\rho'_i} \left(1 - \frac{x_i^{\mu_i}}{t_i^{\mu_i}}\right)^{\sigma_i'},\ i = 1, \ldots, r;$$

and the functions $H[x_1, \ldots, x_r]$ and $H_1[x_1, \ldots, x_r]$ are the same as defined in (2.5) above.

The fractional integral operators (3·2) and (3.3) are valid under the following set of conditions : for $i = 1, \ldots, r$,

(*i*) $\quad 1 \leqslant p_i, q_i < \infty, \dfrac{1}{p_i} + \dfrac{1}{q_i} = 1;$

(*ii*) $\quad \displaystyle\sum_{i=0}^{r} [\text{Re}\,(\eta_i) + k_i\, \rho_i\, \tau^{(i)}] > \max\left(-\frac{1}{q_i}, -\frac{1}{p_i}\right),$

(*iii*) $\quad \displaystyle\sum_{i=1}^{r} [\text{Re}(k_i a_i) + k_i \sigma_i\, \tau^{(i)}] > \max\left(-\frac{1}{q_i}\right),$

$$\text{(iv)}\quad \sum_{i=1}^{r} [\operatorname{Re}(\mu_i a_i) + \mu_i \sigma'_i T^{(i)}] > \max\left(-\frac{1}{q_i}\right),$$

$$\text{(v)}\quad \sum_{i=1}^{r} [\operatorname{Re}(\eta_i) + \mu_i \rho'_i \tau^{(i)}) > \max\left(-\frac{1}{q_i}, -\frac{1}{p_i}\right),$$

$$\text{(vi)}\quad f(x_1, \ldots, x_r) \in L_{p_i}(0, \infty),\ i = 1, \ldots, r;$$

$$\text{where}\quad T^{(i)} = \min_{1 \leqslant j \leqslant m^{(r)}} [\operatorname{Re}(f_j^{(i)}/F_j^{(i)})]$$

4. MAIN RESULTS

Theorem 1. If $R[f(x)] = R\begin{bmatrix} \eta, a;\ k & \\ & ; f(x) \\ a_1, \ldots, a_r;\ x & \end{bmatrix}$ be given by (2·1), then

$$(4{\cdot}1)\qquad R[H[b_1 x^{\beta_1}, \ldots, b_r x^{\beta_r}]$$

$$= H^{0,\ \varepsilon+\varepsilon_1+2\ :\ P}_{A+A_1+2,\ C+C_1+1\ :\ Q}\left(\begin{matrix} B_1, P_1 : Q_1; \\ B_2, P_2 : Q_2; \end{matrix}\ b_1 x^{\beta_1}, \ldots, b_r x^{\beta_r}, a_1, \ldots, a_r\right)$$

where P, Q, B_1, B_2, P_1, P_2, Q_1, Q_2 stand for the parameters indicated below :

$$P : (u', v'); \ldots; (u^{(r)}), (v^{(r)}); (m', n'); \ldots; (m^{(r)}, n^{(r)}),$$

$$Q : [B', D']; \ldots; [B^{(r)}, D^{(r)}] : [M', N']; \ldots; [M^{(r)}, N^{(r)}],$$

$$B_1 : \left[1 - \frac{\eta+1}{k} : \frac{\beta_1}{k}, \ldots, \frac{\beta_r}{k}, \rho, \ldots, \rho\right] [-a : 0, \ldots, 0, \sigma, \ldots, \sigma],$$

$$B_2 : \left[-a - \frac{\eta+1}{k} : \frac{\beta_1}{k}, \ldots, \frac{\beta_r}{k}\ \rho+\sigma, \ldots, \rho+\sigma\right],$$

$$P_1 : [(a) : \theta', \ldots, \theta^{(r)}, 0, \ldots, 0]; [(u) : 0, \ldots, 0, \lambda', \ldots, \lambda^{(r)}],$$

$$P_2 : [(c) : \psi', \ldots, \psi^{(r)}, 0, \ldots, 0]; [(v) : 0, \ldots, 0, \gamma', \ldots, \gamma^{(r)}],$$

$$Q_1 : [(b'), \phi']; \ldots; [(b^{(r)}, \phi^{(r)}]; (e'), E']; \ldots; [(e^{(r)}), E^{(r)}],$$

$$Q_2 : [(d'), \delta']; \ldots; [(d^{(r)}), \delta^{(r)}]; [(f'), F']; \ldots; [(f^{(r)}), F^{(r)}],$$

and (4.1) is valid under conditions obtainable from those given earlier for [2.1).

Theorem 2. If $k[f(x)] = K\begin{bmatrix} \eta, a;\ \mu & \\ a_1, \ldots, a_r;\ x & ; (x) \end{bmatrix}$ be given by (2.2), then

$$(4.2)\qquad K[H_1[b_1/x^{\beta_1}, \ldots, b_r/x^{\beta_r}]$$

$$H^{0,\,2\ \ :\ P}_{A+A_1+2,\ C+C_1+1\ :\ Q}\left(\begin{matrix} B_3,\ P_1 : Q_1; \\ B_4,\ P_2 : Q_2; \end{matrix}\ b_1/x^{\beta_1},\ \ldots,\ b_r/x^{\beta_r},\ a_1,\ \ldots,\ a_r\right)$$

where

$$B_3 : \left[1-\frac{\eta}{\mu};\ \frac{\beta_1}{\mu},\ \ldots,\frac{\beta_r}{\mu},\ \rho',\ \ldots,\ \rho'\right],\ [-a : 0,\ \ldots,\ 0,\ \sigma',\ \ldots,\ \sigma'],$$

$$B_4 : \left[-a-\frac{\eta}{\mu} : \frac{\beta_1}{\mu},\ \ldots,\ \frac{\beta_r}{\mu},\ \rho'+\sigma',\ \ldots,\ \rho'+\sigma'\right],$$

and P, Q, P_1, P_2, Q_1, Q_2 are given wi·h (4.1).

(4.2) is valid under conditions obtainable from those given for (2.2).

Theorem 3. If $R^*\,[f(x)]=R^*\left[\begin{matrix}\eta_i,\ a_i;\ k_i; \\ a_i;\ x_i\end{matrix}\ f(x_1,\ \ldots,\ x_r)\right]$ be given by (3.1), then

(4.3) $$R^*\,[H[b_1x_1^{\beta_1},\ \ldots,\ b_rx_r^{\beta_r}]]$$

$$=H^{0,\ \varepsilon+\varepsilon_1+r\ \ :\ E}_{A+A_1+r,\ C+C_1+r\ :\ F}\left(\begin{matrix} B_5,\ P_1 : Q_3; \\ B_6,\ P_2 : Q_4; \end{matrix}\ b_1x_1^{\beta_1},\ \ldots,\ b_rx_r^{\beta_r},\ a_1,\ \ldots, a_r\right)$$

where E, F, B_5, B_6, P_1, P_2, Q_3, Q_4 stand for the parameters as follows :

$$E : (u',\ v');\ \ldots;\ (u^{(r)},\ v^{(r)});\ (m',\ n'+1);\ \ldots;\ (m^{(r)},\ n^{(r)}+1),$$

$$F : [B',\ D'];\ \ldots;\ [B^{(r)},\ D^{(r)}];\ [M'+1,\ N'];\ [M^{(r)}+1,\ N^{(r)}],$$

$$B_5 : \left[1-\frac{\eta_j+1}{k_j} : \frac{\beta_j}{k_j},\ 0,\ \ldots,\ 0,\ \rho_j,\ 0,\ \ldots,\ 0\right]_{j=1,\ \ldots,\ r}$$

$$B_6 : \left[-a_j-\frac{\eta_j+1}{k_j} : \frac{\beta_j}{k_j},\ 0,\ \ldots,\ 0,\ \rho_j+\sigma_j,\ 0,\ \ldots,\ 0\right]_{j=1,\ \ldots,\ r}$$

$$P_1 : [(a) : \theta',\ \ldots,\ \theta^{(r)},\ 0,\ \ldots,\ 0],\ (u) : 0,\ \ldots,\ 0,\ \lambda',\ \ldots,\ \lambda^{(r)}],$$

$$P_2 : [(c) : \psi',\ \ldots,\ \psi^{(r)},\ 0,\ \ldots,\ 0],\ [(v) : 0,\ \ldots,\ 0,\ \gamma',\ \ldots,\ \gamma^{(r)}],$$

$$Q_3 : [(b'),\ \phi'];\ \ldots;\ [(b^{(r)}),\ \phi^{(r)}];\ [-a,\ \sigma_1],\ [(e'),\ E'];\ \ldots;\ [-a_r,\ \sigma_r],\ [(e^{(r)}),\ E^{(r)}],$$

$$Q_4 : [(d'),\ \delta'];\ \ldots;\ [(d^{(r)}),\ \delta^{(r)}];\ [(f'),\ F'];\ \ldots;\ [f^{(r)},\ F^{(r)}].$$

The conditions of validity are obtainable from those given with (3.1).

Theorem 4. If $K^*[f(x_i)]=K^*\left[\begin{matrix}\eta_i,\ a_i;\ \mu_i; \\ a_i;\ x_i\end{matrix};\ f(x_1,\ \ldots,\ x_r]\right]$ be given by (3.2), then

(4.4) $$K^*[H_1[b_1/x_1^{\beta_1},\ \ldots,\ b_r/x_r^{\beta_r}]]$$

$$H^{0,\ r\ \ :\ E}_{A+A_1+r,\ C+C_1+r\ :\ F}\left(\begin{matrix} B_7,\ P_1 : Q_3; \\ B_8,\ P_2 : Q_4; \end{matrix}\ b_1/x_1^{\beta_1},\ \ldots,\ b_r/x_r^{\beta_r},\ a_1,\ \ldots,\ a_r\right)$$

where B_7, B_8 stand for

$$B_7 : \left[1-\frac{\eta_j}{\mu_j} : \frac{\beta_j}{\mu_j}, 0, \ldots, 0, \rho'_j, 0, \ldots, 0 \right]_{j=1, \ldots, r'}$$

$$B_8 : \left[-\sigma_j-\frac{\eta_j}{\mu_j} : \frac{\beta_j}{\mu_j}, 0, \ldots, 0, \rho'_j+\sigma_j', 0, \ldots, 0 \right]_{j=1, \ldots, r'}$$

and E, F, P_1, P_2, Q_3, Q_4 are given with (4·3), provided that the conditions obtainable from those given with (3·2) hold.

Outline of Proofs : To prove (4·1), we use (2·1), express the two multivariable H-functions in terms of Mellin-Barnes type contour integrals by virtue of (2·5) and (1·1), change the order of integration (which is justified, under the conditions stated, due to the absolute convergence of the integrals involved in the process), evaluate the inner x-integral and finally interpret, by the definition of the H-function of $2r$ variables; we thus obtain the desired result on the right-hand side of (4·1).

Theorems 2, 3 and 4 are established with the help of (2·2), (3·1) and (3·2), respectively, and on proceeding on similar lines as indicated in the proof of (4·1) above.

Several particular cases of Theorems 1 to 4 can be given (see [1, Chapter 12]).

ACKOWLEDGEMENTS

The authors are thankful to Proof. M. C. Gupta for his kind guidance and keen interest in the preparation of this paper. The authors are highly grateful to Prof. H. M. Srivastava for his valuable suggestions which led to considerable improvement of this paper.

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Jñānabha, Vol, 20, 1990

Perturbations of the Elements of Near-Circular Earth Satellite Orbits

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(Received : August 25, 1988, Revised : November 2, 1989)

ABSTRACT

The expressions are found for perturbations of the elements of near-circular Earth's satellite orbits to an accuracy of the second order of eccentricity when second harmonic of Earth's oblateness is taken. In one of the section the satellite coordinates and perturbed topocentric coordinates are also calculated.

1. The Differential Equations of Motion

The Lagrange's equations in terms of the variables as used in Bhatnagar-Taqvi [3].

$$\frac{da}{dt}=\frac{2}{na}\cdot\frac{\partial R}{\partial\lambda_0},$$

$$\frac{dh}{dt}=\frac{\sqrt{(1-h^2-l^2)}}{na^2}\cdot\frac{\partial R}{\partial l}-\frac{\sqrt{(1-h^2-l^2)}}{na^2}\cdot\frac{h}{1+\sqrt{(1-h^2-l^2)}}\cdot\frac{\partial R}{\partial\lambda_0}$$
$$-\frac{l\cot i}{na^2\sqrt{(1-h^2-l^2)}}\cdot\frac{\partial R}{\partial i},$$

$$\frac{dl}{dt}=-\frac{\sqrt{(1-h^2-l^2)}}{na^2}\frac{\partial R}{\partial h}-\frac{\sqrt{(1-h^2-l^2)}}{ha^2}\cdot\frac{1}{1+\sqrt{(1-h^2-l^2)}}\frac{\partial R}{\partial\lambda_0}$$
$$+\frac{h\cot i}{na^2\sqrt{(1-h^2-l^2)}}\cdot\frac{\partial R}{\partial i},$$

$$\frac{d\Omega}{dt}=\frac{\operatorname{cosec} i}{na^2\sqrt{(1-h^2-l^2)}}\cdot\frac{\partial R}{\partial i},$$

$$\frac{di}{dt}=\frac{\cot i}{na^2\sqrt{(1-h^2-l^2)}}\left(l\frac{\partial R}{\partial h}-h\frac{\partial R}{\partial l}+\frac{\partial R}{\partial\lambda_0}\right)-\frac{\operatorname{cosec} i}{na^2\sqrt{(1-h^2-l^2)}}\cdot\frac{\partial R}{\partial\Omega},$$

$$\frac{d\lambda_0}{dt}=-\frac{2}{na}\cdot\frac{\partial R}{\partial a}+\frac{\cot i}{na^2\sqrt{1-h^2-l^2)}}\cdot\frac{\partial R}{\partial i}+\frac{\sqrt{(1-h^2-l^2)}}{1+\sqrt{(1-h^2-l^2)}}\cdot\frac{1}{na^2}\left(h\frac{\partial R}{\partial h}\right.$$
$$\left.+l\frac{\partial R}{\partial l}\right) \quad ...(1)$$

For circular orbits the corresponding equations can be obtained from (1) by putting $h=0$, $l=0$.

2. The Expansion of the Disturbing Function

Writing the potential of the Earth at an external point Sterne [1], the disturbing function R to the 2*nd* power of the flattening is given by Proskurin and Batrakov [4].

$$R=-\frac{J}{3}fm\left(\frac{a'^2}{r^3}\right).(3\sin^2\delta-1)+\frac{D}{35}fm\left(\frac{a'^4}{r^5}\right)(35\sin^4\delta-30\sin^2\delta+3) \quad \text{...(2)}$$

where f is the gravitational constant, m the mass of the Earth, r the radius vector of the satellite, the equitorial radius of the Earth, δ the inclination of the satellite, J and D are constants depending on the oblateness of the Earth.

Substituting the value of $\sin\delta$ in (2) Bhatnagar-Taqvi [3] and taking only the contribution of the second power of oblateness in R we have (the contribution of the first power of oblateness in R has already been calculated in Bhatnagar-Taqvi [3])

$$\begin{aligned}R=\frac{1}{35\times 8}D\;fm\left(\frac{a'^4}{a^5}\right)\Big[&(105\sin^4 i-120\sin^2 i+24)\left(\frac{a}{r}\right)^5-20(7\sin^4 i\\&-6\sin^2 i)\cos 2\omega\left(\frac{a}{r}\right)^5\cos 2v+20\,(7\sin^4 i-6\sin^2 i)\sin 2w\\&\left(\frac{a}{r}\right)^5\sin 2v+35\sin^4 i\times\cos 4w\left(\frac{a}{r}\right)^5\cos 4v-35\sin^4 i\sin 4w\\&\left(\frac{a}{r}\right)^5\sin 4v\Big]. \qquad \text{...(3)}\end{aligned}$$

where a=semi-major axis of the orbit.

Calculating the values of $\left(\frac{a}{r}\right)^5$, $\left(\frac{a}{r}\right)^5\cos 2v$, $\left(\frac{a}{r}\right)^5\sin 2v$, $\left(\frac{a}{r}\right)^5\cos 4v$ and $\left(\frac{a}{r}\right)^5\sin 4v$ in terms of mean anomaly M and taking terms upto the 3*rd* power of e, we have

$$\left(\frac{a}{r}\right)^5=1+5e^2+\left(5e+\frac{135}{8}e^3\right)\cos M+10e^2\cos 2M+\frac{145}{3}e^3\cos 3M$$

$$\begin{aligned}\left(\frac{a}{r}\right)^5\cos 2v=\frac{3}{4}e^2+\left(\frac{e}{2}+\frac{37}{12}e^3\right)\cos M+(1+e^2)\cos 2M\\+\left(\frac{9}{2}e-\frac{3}{16}e^3\right)\cos 3M+\frac{53}{4}e^2\cos 4M\\+\frac{1541}{48}e^3\cos 5M.\end{aligned}$$

Similarly the values of $\left(\frac{a}{r}\right)^5 \cos 4v$, $\left(\frac{a}{r}\right)^5 \sin 2v$ and $\left(\frac{a}{r}\right)^5 \sin 4v$ can be calculated.

Substituting the above values in (3) and changing M, w, e in terms of new variables l, h etc., we have

$$R = R_1 + R_2 + R_3 + \ldots$$

where

$$R_1 = \frac{1}{35 \times 8}\, \mathrm{D}.\, fm.\, \frac{a'^4}{a^5}\, (105 \sin^4 i - 120 \sin^2 i + 24)\, [1 + 5(l \cos \lambda$$

$$+ h \sin \lambda)] - \frac{1}{14} . \mathrm{D}.\, fm.\, \frac{a'^4}{a^5}\, (7 \sin^4 i - 6 \sin^2 i)\, [\tfrac{1}{2}(l \cos \lambda - h \sin \lambda)$$

$$+ \cos 2\lambda + \frac{9}{2}(l \cos 3\lambda + h \sin 3\lambda)] + \tfrac{1}{8}\, \mathrm{D}.\, fm.\, \frac{a'^4}{a^5} \sin^4 i \Big[-\frac{3}{2}$$

$$(l \cos 3\lambda - h \sin 3\lambda) + \cos 4\lambda + \frac{13}{2}\, (l \cos 5\lambda + h \sin 5\lambda) \Big].$$

Similarly the values of R_2 and R_3 can be calculated.

3. Perturbations of the Orbital Elements

In order to calculate the perturbations of the orbital elements of a satellite, we will follow the same technique as Bhatagar—Taqvi [3]. The results are shown in equation (4) where constants of integration are not mentioned. Constants of integration can be calculated as in our earlier paper [3].

$$\delta a = \frac{1}{28}\, D.\, \frac{a'^4}{a^3}\, (105 \sin^4 i - 120 \sin^2 i + 24)\, [l \cos \lambda + h \sin \lambda]$$

$$- \frac{1}{7}\, D.\, \frac{a'^4}{a^8}\, (7 \sin^4 i - 6 \sin^2 i) \Big[\frac{1}{2}(l \cos \lambda - h \sin \lambda) + \cos 2\lambda$$

$$+ \frac{9}{2}(l \cos 3\lambda + h \sin 3\lambda) \Big] + \frac{1}{8}\, D.\, \frac{a'^4}{a^3} \sin^4 i\, [-3(l \cos 3\lambda$$

$$- h \sin 3\lambda) + 2 \cos 4\lambda + 13\, (l \cos 5\lambda + h \sin 5\lambda)]$$

$$+ \frac{1}{14}\, D\, \frac{a'^4}{a^3}\, (105 \sin^4 i - 120 \sin^2 i + 24)\, [(l^2 - h^2) \cos 2\lambda]$$

$$- \frac{1}{7}\, D\, \frac{a'^4}{a^3}\, (7 \sin^4 i - 6 \sin^2 i) \Big[(l^2 + h^2) \cos 2\lambda$$

$$+ \frac{53}{4} \{(l^2 - h^2) \cos 4\lambda + 2hl \sin 4\lambda\} \Big]$$

$$+ \frac{1}{4}\, D\, \frac{a'^4}{a^3} \sin^4 i\, [\tfrac{1}{2}(l^2 - h^2) \cos 2\lambda - h \sin 2\lambda - 11\, (l^2 + h^2) \cos 4\lambda$$

$$+ \frac{51}{2} \{(l^2 - h^2) \cos 6\lambda + 2hl \sin 6\lambda\}]. \qquad \ldots(4a)$$

$$\begin{aligned}
\delta\Omega = {} & \frac{3}{14} D \left(\frac{a'}{a}\right)^4 (7 \sin^2 i - 4) \cos i (nt + 5\, (l \sin \lambda - h \cos \lambda)] \\
& - \frac{2}{7} D \left(\frac{a'}{a}\right)^4 (7 \sin^2 i - 3) \times \cos i \left[\tfrac{1}{2}(l \sin \lambda + h \cos \lambda)\right. \\
& \left. + \tfrac{1}{2} \sin 2\lambda + \tfrac{3}{2}\, (l \sin 3\lambda - h \cos 3\lambda)\right] \\
& + \tfrac{1}{2} D \left(\frac{a'}{a}\right)^4 \sin^2 i \cos i \left[-\tfrac{1}{2}(l \sin 3\lambda + h \cos 3\lambda) + \tfrac{1}{4} \sin 4\lambda\right. \\
& \left. + \frac{13}{10} (l \sin 5\lambda - h \cos 5\lambda)\right] + \frac{3}{14} D \left(\frac{a'}{a}\right)^4 (7 \sin^2 i - 4) \cos i \left[\frac{11}{2} (h^2\right. \\
& \left. + l^2) nt + 5\, (l^2 - h^2) \sin 2\lambda \right] - \frac{2}{7} D \left(\frac{a'}{a}\right)^4 (7 \sin^2 i - 3) \cos i \left[\frac{3}{4}\right. \\
& (l^2 - h^2) nt + \frac{3}{4} (l^2 + h^2) \sin 2\lambda + \frac{53}{16} \left.\{(l^2 - h^2) \sin 4\lambda - 2hl \cos 4\lambda\}\right] \\
& + \tfrac{1}{2} D \left(\frac{a'}{a}\right)^4 \sin^2 i \cos i \left[\tfrac{1}{4}(l^2 - h^2) \sin 2\lambda + \tfrac{1}{2}\, hl \cos 2\lambda\right. \\
& \left. - \frac{21}{8} (l^2 + h^2) \sin 4\lambda + \frac{17}{4} \{l^2 - h^2) \sin 6\lambda - 2hl \cos 6\lambda\}\right]. \qquad \text{...(4b)}
\end{aligned}$$

$$\begin{aligned}
\delta i = {} & -\frac{1}{14} D \left(\frac{a'}{a}\right)^4 \cos i\, (7 \sin^3 i - 6 \sin i) \left[l \cos \lambda - h \sin \lambda + \cos 2\lambda\right. \\
& \left. + \tfrac{1}{2}(h^2 + l^2) \cos 2\lambda + 3(l \cos 3\lambda + h \sin 3\lambda)\right] \\
& + \tfrac{1}{8} D \left(\frac{a'}{a}\right)^4 \sin^3 i \cos i \left[2(h \sin 3\lambda - l \cos 3\lambda)\right. \\
& \left. + \frac{26}{5} (l \cos 5\lambda + h \sin 5\lambda) + \cos 4\lambda + \tfrac{1}{2}(h^2 + l^2) \cos 4\lambda\right] \\
& + \frac{1}{35 \times 8} D \left(\frac{a'}{a}\right)^4 (105 \sin^4 i - 120 \sin^2 i + 24) \cot i \left[-20\, hl \sin 2\lambda\right. \\
& \left. + 10\, (l^2 - h^2) \cos 2\lambda\right] - \frac{1}{14} D \left(\frac{a'}{a}\right)^4 (7 \sin^3 i - 6 \sin i) \cos i \left[-3hlnt\right. \\
& \left. + \frac{53}{4} hl \sin 4\lambda + \frac{53}{8} (l^2 - h^2) \cos 4\lambda + (l^2 + h^2) \cos 2\lambda\right. \\
& + \tfrac{1}{8} D \left(\frac{a'}{a}\right)^4 \sin^3 i \cos i \left[-2lh \sin 2\lambda + (l^2 - h^2) \cos 2\lambda\right. \\
& \left. - 11(l^2 + h^2) \cos 4\lambda - 34hl \sin 6\lambda - 17\, (h^2 - l^2) \cos 6\lambda\right]. \qquad \text{...(4c)}
\end{aligned}$$

Similarly we can calculate the perturbations of the orbital elements δh, δl and $\delta\lambda_0$.

For circular orbits the perturbations can be obtained by putting $h=0$, $l=0$.

4. Calculating the Geocentric Satellite Coordinates Taking Terms Upto the Second Order of *h*, *l* and Corrections of the Topocentric Satellite Coordinates

In order to calculate λ, we will use the same technique as in our earlier paper [3], we have

$$
\begin{aligned}
\lambda={}&\lambda_0+n_0\,(t-t_0)-3J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\sin^2 i\right)(l\sin\lambda-h\cos\lambda)\\
&-\frac{3}{2}\left(\frac{a'}{a}\right)^2 J\sin^2 i\Big[-\frac{1}{2}\,l\sin\lambda-\frac{1}{2}\,h\cos\lambda+\frac{1}{2}\sin 2\lambda\\
&+\frac{7}{6}l\sin 3\lambda-\frac{7}{6}h\cos 3\lambda\Big]-J\left(\frac{a'}{a}\right)^2\left(1-\frac{3}{2}\sin^2 i\right)\Big[\frac{9}{4}(l^2\\
&-h^2)\sin 2\lambda-\frac{9}{2}lh\cos 2\lambda\Big]-\tfrac{3}{2}J\left(\frac{a'}{a}\right)^2\sin^2 i\times\Big[-\frac{5}{4}(h^2+l^2)\sin 2\lambda\\
&+\frac{17}{8}(l^2-h^2)\sin 4\lambda-\frac{17}{4}\,hl\cos 4\lambda\Big]+\frac{1}{35\times 8}D\,(105\sin^4 i\\
&-120\sin^2 i+24)\,[5\{2+11\,(l^2+h^2)\}nt+\frac{75}{2}\,(l\sin\lambda-h\cos\lambda)\\
&+40\,(l^2-h^2)\sin 2\lambda]-\frac{1}{14}D\left(\frac{a'}{a}\right)^4(7\sin^4 i-6\sin^2 i)\Big[\frac{15}{4}(l\sin\lambda\\
&+h\cos\lambda)+\frac{7}{2}\sin 2\lambda+\frac{45}{4}\,(l\sin 3\lambda-h\cos 3\lambda)+\frac{33}{4}(l^2+h^2)\,nt\\
&-6(l^2+h^2)\sin 2\lambda+\frac{795}{32}\{(l^2-h^2)\sin 4\lambda-2hl\cos 4\lambda\}\Big]\\
&+\frac{1}{8}\cdot D\left(\frac{a'}{a}\right)^4\sin^4 i\times\Big[-\frac{15}{4}\,(l\sin 3\lambda+h\cos 3\lambda)+\frac{7}{4}\sin 4\lambda\\
&+\frac{39}{4}\,(l\sin 5\lambda-h\cos 5\lambda)+2(l^2-h^2)\sin 2\lambda+4hl\cos 2\lambda-22\,(h^2\\
&+l^2)\sin 4\lambda-\frac{119}{2}\,\{(l^2-h^2)\sin 6\lambda-2hl\cos 6\lambda\}\Big]\\
&-\frac{3}{14}D\left(\frac{a'}{a}\right)^4(7\sin^2 i-4)\cos^2 i\Big[\Big\{1+\frac{11}{2}\,(l^2+h^2)\Big\}nt\\
&+5(l\sin\lambda-h\cos\lambda)+5(l^2-h^2)\sin 2\lambda\Big]\\
&+\frac{2}{7}D\left(\frac{a'}{a}\right)^4(7\sin^2 i-3)\cos^2 i\times[\tfrac{1}{2}(l\sin\lambda+h\cos\lambda)+\tfrac{1}{2}\sin 2\lambda\\
&+\frac{3}{2}\,(l\sin 3\lambda-h\cos 3\lambda)+\frac{3}{4}\,(l^2-h^2)\,nt+\frac{3}{4}\,(l^2+h^2)\sin 2\lambda\\
&+\frac{53}{16}\{(l^2-h^2)\sin 4\lambda-2hl\cos 4\lambda\}]-\tfrac{1}{2}D\left(\frac{a'}{a}\right)^4\cos^2 i\sin^2 i[-\tfrac{1}{2}(l\sin 3\lambda
\end{aligned}
$$

$$+h\cos 3\lambda)+\tfrac{1}{4}\sin 4\lambda+\tfrac{13}{16}\,l\sin 5\lambda-h\cos 5\lambda-\frac{21}{8}(l^2+h^2)\sin 4\lambda$$
$$+\frac{1}{4}\,(i^2-h^2)\,\sin\,2\lambda+\frac{1}{2}\,hl\cos 2\lambda+\frac{17}{4}\,\{(l^2-h^2)\sin\,6\lambda$$
$$-2h\,l\cos\,6\lambda\}].$$

In order to calculate geocentric coordinates, we can apply the same technique as in earlier paper and finally the topocentric coordinates and hence the corrections in topocentric coordinates can be written as [3, *p.* 443–444].

$$\Delta x'=\frac{\Delta x}{r}\Delta a-\left(\frac{ax}{r}\cos\lambda+\frac{2}{\cos i}y\sin\lambda\right)\Delta l-\left(\frac{ax}{r}\sin\lambda\right.$$
$$\left.-\frac{2}{\cos i}\,y\cos\lambda\right)\Delta h+\left(\frac{Bax}{r}-\frac{Cy}{\cos i}\right)\Delta\lambda-y\Delta\Omega+\frac{xA'}{2r}\Delta a$$
$$+\left\{\frac{ax}{r}\,(l-l\cos 2\lambda-h\sin 2\lambda)-\frac{5}{2}\cdot\frac{y}{2\cos i}\,(l\sin 2\lambda-h\cos 2\lambda)\right\}\Delta l$$
$$+\left\{\frac{ax}{r}(h+h\cos 2\lambda-l\sin 2\lambda)+\frac{5}{2}\,\frac{y}{\cos i}\,(h\sin 2\lambda+l\cos 2\lambda)\right\}\Delta h$$
$$+\left(\frac{ax}{r}\,B'-\frac{5}{2}\cdot\frac{yC'}{\cos i}\right)\qquad\ldots(5)$$

and similarily the values of $\Delta y'$ and $\Delta z'$ can be calculated where

$A=1-l\cos\lambda-h\sin\lambda$; $B=l\sin\lambda-h\cos\lambda$; $C=1+2l\cos\lambda+2h\sin\lambda$,
$A'=(h^2+l^2)-(l^2-h^2)\cos 2\lambda-2hl\sin 2\lambda$;
$B'=(l^2-h^2)\sin 2\lambda-2hl\cos 2\lambda$.
$C'=(l^2-h^2)\cos 2\lambda+2hl\sin 2\lambda$.

The corresponding expressions for $\Delta x'$, $\Delta y'$ and $\Delta z'$ for circular orbits can be obtained by putting $A=1=C$, $B=0$, $r=a$ and $A'=B'=C'=0$ and $\Delta\lambda=\Delta\lambda_0-\frac{3}{2}\cdot\frac{n}{a}\Delta a.\,t.$ If the corrections of the orbital elements Δa, Δl, Δh, $\Delta\Omega$, Δi and $\Delta\lambda_0$ and the geocentric coordinates x, y and z in a moving coordinate system is known then equations (5) will give us the corrections of topocentric rectangular satellite coordinates.

The results proved in this paper are to an accuracy of second order of eccentricity. These results can be used not only for near-circular Earth Satellite orbits but can be used for elliptical orbits of small eccentricity.

Example : For satellite 1958 (Vanguard I), the orbital data are (McCuskey, 1960) [5]

$P=134.04$ min.,	$\omega=162°.7$,	$\Omega=318°.8$
$e=0.190$,	$a=8676$ Km.	
$i=34.25$,	$\tau=7.5188$ January 1960.	

We wish to calculate the perturbations and topocentric coordinates of the satellite after six months at a place whose latitude is 28°.40 N. we know that

a'=equatorial radius=6378.39 Km.
J=First power of earth's oblateness$=1.637\times10^{-3}$

$D=10.6\times10^{-6}$

Further, we have

$\pi_0=\omega+\Omega=481^\circ.5$

$h_0=e\sin(\pi-\Omega)=e\sin\omega=0.0565003$

$l_0=e\cos(\pi-\Omega)=e\cos\omega=0.1814044$

$P_0=134.04$ min. $=0.093083$ days

$$n_0=\frac{2\pi}{P}=\frac{2\pi}{0.093083}\ \text{rad/day}$$

$\tau=t_0\quad=7.5188$

$\varepsilon_0=\pi+n\,(t_0-\tau)=481^\circ.5$

$\lambda_0{}^\circ=(\lambda_0)_{t=t_0}=\varepsilon-\Omega=162^\circ.7$

and

$\lambda_0=\lambda_0{}^\circ+n_0(t-t_0)=268^\circ.180$

All the calculations have been worked on computer *IBM*—360/44. Further, we have taken our readings after 30 days intervals. We have studied perturbations for a large number of orbits during six months. Here, we shall mention perturbations only after 30 days and 180 days.

Case 1.Purturbations after 30 days.

Elements	*Perturbations due to the effect of J*	*Perturbations due to the effect of J+D*	*Perturbations due to the effect of D*
$a=a_0+[\delta a-(\delta a)_{t=t_0}]$	8673·866573	8673·873415	0·0006842 *Km.*
$\Omega=\Omega_0+[\delta\Omega-(\delta\Omega)_{t=t_0}]$	3·97885278	3·978034086	−0·000851192 *rad.*
$i=i_0+[\delta i-(\delta i)_{t=t_0}]$	0·5977167567	0·5976921089	− 0·00002465 *rad.*
$h=h_0+[\delta h-(\delta h)_{t=t_0}]$	0·448541649	0·448842864	0·0003012 6
$l=l_0+[\delta l-(\delta l)_{t=t_0}]$	0·059591046	0·059588240	− 0·000002807 „
$\lambda_0=\lambda_0^0+[\delta\lambda_0-(\delta\lambda_0]_{t=t_0}]$	6·318054544	6·315006551	−0·003047993 *rad.*
$e=$	0·45248282	0·45278105	0·00029823
$\omega=$	1·438714703	1·438808459	0·000093756 *rad.*
$x=$	3934·731799	3935·326893	0·595094 *Km.*
$y=$	−9453·547539	−9455·940758	2·393219 *Km.*
$z=$	−6435·881862	−6437·170058	1·288196 *Km.*
$x'=$	−290·9203478	−293·4647838	2·544436 *Km.*
$y'=$	−5763·358846	−5769·350244	5·991368 *Km.*
$z'=$	−9470·717992	−9472·006188	1·288196 *Km.*

Case II : Perturbations after 180 days :

Elements	*Perturbations due to the effect of J*	*Perturbations due to the effect of J+D*	*Perturbations due to the effect of D*
$a=$	8673.047088	8673.057840	0.010752 km.
$\Omega=$	−3.96360774	−3.97010357	0.00649583 rad.
$i=$	0.597613411	0.597467354	−0.00014605 rad.
$h=$	2.41248581	2.41429178	0 00180597
$l=$	−0.55181004	−0.55182689	0.00001685
$\lambda_0=$	23.71129616	23.69299935	−0.01829681 rad.
$e=$	2.47478931	2.47655360	0.00176429
$\omega=$	−1.34593362	−1.34608958	0 00015596 rad.
$x=$	−38760.01128	−38619.43959	140.57161 km.
$y=$	−17163.15114	−17474.68727	311.53613 km.
$z=$	−11681.90826	−11890.21753	208.30927 km.
$x'=$	−42062.22428	−41892.12256	−170.10172 km.
$y'=$	−21698.45907	−22031.34996	332.89089 km.
$z'=$	−14716.74439	−14925.05366	208.30927 km.

Here the semi-major axis is given in kilometers and in the secular perturbations of the elements e, i, Ω, π, and λ_0, the time must be taken in mean solar days.

We may note that in the case of semi-major axis, the contribution of D increases and becomes significant as the value of a decreases and the value of e increases. The contributions of D after 30 days and 180 days are 0.0006842 and 0.010752 when the value of e increases from 0.00029823 to 0.00176429 which is quite significant and hence we cannot ignore the contribution of D.

We note that in the case of orbital inclination of the satellite, the contribution of D increses and becomes significant as the value of e increases. The contributions of D after 30 days and 180 days are −0.00002465 and −0.00014605 when the value of e increses from 0·00029823 to 0.00176429. The contribution of D is quite significant after 360 days where the contribution of D is −0.00029044 when $e=0.00349655$.

As the value of a decreases and the value of *e* increases, the geocentric rectangular coordinates (x, y, z) of the satellite vary slowly but it has significant values after 180 days. The contribution of *D* after 30 days and 180 days are 0.595094, 2.393219, 1.288196 and 140.57161, 311,53613, 208.30927 respectively, which is quite significant.

Further, we may note that when the value of *e* increases and the value of a decreases, the topocentric coordinates vary very frequently. The contributions of *D* after 30 days 180 days are 2.544436, 5.991398, 1.288196 and —170.10172, 332.89089, 208.30927 when the value of *e* increases from 0.00029823 to 0.00176429 and the value of a decreases from 0.0006842 to 0.010752. Hence we note that there are significant values of *D* and its contributions cannot be ignored.

Acknowledgements

I am very thankful to Prof. H. M. Srivastava for his valuable suggestions and guidance for the preparation of this paper. I am also grateful to him for his keen interest in this paper.

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Jñānābha, Vol, 20, 1990

Stability of Stratified Fluids in Porous Media with Variable Fluid Viscosity and Medium Permeability

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(Recieved : October 21, 1988, Revised : August 8, 1989)

ABSTRACT

The general equation governing the stability of a stratified fluid in a stratified porous medium has been derived. Assuming an exponential stratifications in density, viscosity, porosity and medium permeability; the dispersion relation has been obtained. For disturbances of all wave numbers, the system is found to be stable for stable stratification and unstable for unstable stratification. With the increase in medium permeability the growth rates increase for certain wave numbers and decrease for the other wave numbers. However the growth rate of unstable modes decrease with the increase in kinematic viscosity.

1. INTRODUCTION

A comprehensive account of the Rayleigh-Taylor instability under varying assumptions of hydrodynamics has been given by Chandrasekhar [1]. Gupta [2] has investigated the stability of a horizontal layer of a perfectly conducting fluid, with continuous density and viscosity stratification in the presence of a horizontal magnetic field (constant as well as variable). The medium has been considered to be non-porous.

The flow through porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicists. A macroscopic equation which describes incompressible flow of Newtonian fluid of viscosity μ through a macroscopically homogeneous and isotropic medium of permeability k_1 is the well known Darcy's equation. The usual viscous term in the equations of fluid motion is replaced by the resistance term $-\left(\frac{\mu}{k_1}\right)\vec{q}$, where $\vec{q}$ is the filter velocity of the fluid. The fluid viscosity and medium permeability are considered to be uniform. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [3].

More often than not, in naturally occurring situations of stratified fluids in porous media in Geophysics, the fluid viscosity and medium permeability are not constants but vary with depth of layer. The present paper therefore, deals with the stability of stratified fluid in porous medium with variable fluid viscosity and medium permeability.

2. PERTURBATION EQUATIONS

The initial stationary state whose stability we wish to examine is that of an incompressible fluid of variable density and viscosity arranged in horizontal strata in porous medium of variable permeability and porosity. Let ρ, μ, p and $\vec{q}\,(u, v, w)$ denote respectively the density, viscosity, pressure and velocity of the fluid; g, k_1 and ε stand for acceleration due to gravity, medium permeability and medium porosity respectively. Then the equations of motion and continuity in porous medium are

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\varepsilon}(\vec{q}\cdot\nabla)\vec{q}\right]=-\nabla p-\rho g\vec{\lambda}-\frac{\mu}{k_1}\vec{q}, \qquad \ldots(1)$$

$$\nabla\cdot\vec{q}=0; \qquad \ldots(2)$$

where $\vec{\lambda}=(0, 0, 1)$. Since the density of a particle moving with the fluid remains unchanged, we have

$$\varepsilon\frac{\partial\rho}{\partial t}+(\vec{q}\cdot\nabla)\rho=0. \qquad \ldots(3)$$

Let $\vec{q}\,(u, v, w)$, $\delta\rho$ and δp denote the perturbations in velocity $(0, 0, 0)$, density ρ and pressure p respectively. Then the linearized perturbation equations are

$$\frac{\rho}{\varepsilon}\frac{\partial\vec{q}}{\partial t}=-\nabla\delta p-g\delta\rho\,\vec{\lambda}-\frac{\mu}{k_1}\vec{q}, \qquad \ldots(4)$$

$$\nabla\cdot\vec{q}=0, \qquad \ldots(5)$$

$$\frac{\partial}{\partial t}\delta\rho=-\omega\frac{d\rho}{dz}. \qquad \ldots(6)$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp\,(ik_x x+ik_y y+nt). \qquad \ldots(7)$$

where k_x, k_y are wave numbers along x and y directions,

$$k=\left(k_x^2+k_y^2\right)^{1/2}$$ and n is a complex constant.

Equations (4), (5), and (6), using expression (7), give

$$\frac{\rho}{\varepsilon}\, nu = -ik_x\delta p - \frac{\mu}{k_1}u, \qquad \text{...(8)}$$

$$\frac{\rho}{\varepsilon}\, nv = -ik_y\delta p - \frac{\mu}{k_1}v, \qquad \text{...(9)}$$

$$\frac{\rho}{\varepsilon}\, nw = -D\delta p - \frac{\mu}{k_1}w - g\rho\delta, \qquad \text{...(10)}$$

$$ik_x u + ik_y v + Dw = 0, \qquad \text{...(11)}$$

$$n\delta\rho = -wD\rho, \qquad \text{...(12)}$$

where $D = d/dz$.

Eleminating p between equations (8)—(10) and using equations (11) and (12); we obtain

$$D\left(\frac{\rho n Dw}{\varepsilon} + \frac{\mu}{k_1}Dw\right) - k^2\left(\frac{\rho n w}{\varepsilon} + \frac{\mu}{k_1}w\right) = -\frac{gk^2}{n\varepsilon}(D\rho)\,w. \qquad \text{...(13)}$$

Equation (13), after simlification, can be written as

$$\left(\frac{n\rho}{\varepsilon} + \frac{\mu}{k_1}\right)(D^2 - k^2)w + \left\{\frac{n}{\varepsilon}D\rho - \frac{n\rho D\varepsilon}{\varepsilon^2} + \frac{D\mu}{k_1} - \frac{\mu}{k_1^2}Dk_1\right\}Dw$$

$$+\frac{gk^2\ D\rho)w}{\varepsilon n} = 3. \qquad \text{...(14)}$$

Equaiion (14) is the general equation governing the stability of stratified fluid in a stratified porous medium.

3. THE CASE OF EXPONENTIALLY VARYING STRATIFICATIONS

Assume the stratifications in density, viscosity, medium porosity and medium permeability of the form

$$\rho = \rho_0 e^{\beta z},\ \mu = \mu_0 e^{\beta z},\ \varepsilon = \varepsilon_0 e^{\beta z},\ k_1 = k_{10} e^{\beta z}, \qquad \text{...(15)}$$

where ρ_0, μ_0, ε_0 and k_{10} are constants and so the kinematic viscosity $\nu(=\mu/\rho = \mu_0/\rho_0)$ is uniform from layer to layer. We also assume that $\beta d \ll 1$, *i.e.*, the variation of density at two neighbouring points in the velocity field which is much less than the average density has a negligible effect on the inertia of the fluid.

Using the stratifications of the form (15), equation (14) transforms to

$$(nk_{10} - \nu_0\varepsilon_0)(D^2 - k^2)w + \frac{gk^2k_{10}\beta}{n}w = 0. \qquad (16)$$

The general solution of equation (16) is

$$w = Ae^{m_1 z} + Be^{m_2 z}, \qquad \text{...(17)}$$

where A, B are two arbitrary constants; m_1, m_2 are given by

$$m_{1,2} = \pm k\left[1 - \frac{g\beta k_{10}}{nL}\right]^{1/2}, \qquad \text{...(18)}$$

and

$$L = (nk_{20} - \nu_0 \varepsilon_0)$$

If the fluid is supposed to be confined between two rigid planes at $z=0$ and $z=d$, then the vanishing of w at $z=0$ is satisfied by the choice

$$w = A(e^{m_1 z} - e^{m_2 z}), \qquad \text{...(20)}$$

while the vanishing of w at $z=d$ requires

$$\exp(m_1 - m_2)d = 1, \qquad \text{...(21)}$$

which implies that

$$(m_1 - m_2)d = 2 \text{ is } \pi, \qquad \text{...(22)}$$

where s is an integer.

Inserting the values of m_1, m_2 from (18) in equation (22) and simplifying, we obtain

$$n^2 + \frac{\nu_0 \varepsilon_0}{k_{10}} n - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2 \pi^2} = 0. \qquad \text{.. (23)}$$

Equation (23) is the dispersion relation governing the stability of stratified (exponentially varying density and viscosity) fluid in a stratified (exponentially varying porosity and medium permeability) porous medium.

If $\beta < 0$ (Stable stratification), (23) does not admit of any positive root of n and the system is always stable for disturbances of all wave numbers.

If $\beta > 0$ (unstable stratification), equation (23) has one positive root and so the system is unstable for disturbances of all wave numbers. Let n_0 denotes the positive root of equation (23). Then

$$n_0^2 + \frac{\nu \varepsilon}{k_1} n_0 - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2 \pi^2} = 0, \qquad \text{... (24)}$$

as

$$\nu_0 = \mu_0/\rho_0 = \mu/\rho \text{ and } \varepsilon_0/k_{10} = \varepsilon/k_1.$$

To fined the role of medium permeability and viscosity on the growth rate of unstable modes, we examine the natures of dn_0/dk_1 and $dn_0/d\nu$. Equation (24) gives

$$\frac{dn_0}{dk_1} = -\frac{\left(n_0^2 - \frac{g\beta k^2 d^2}{k^2 d^2 + s^2 \pi^2}\right)}{2n_0 k_1 + \nu\varepsilon}. \qquad \text{..(25)}$$

If $n_0^2 > \frac{g\beta k^2 d^2}{k^2 d^2 + s^2 \pi^2}$, the growth rate decreases with the increase in medium permeability where as if $n_0^2 < \frac{g\beta k^2 d^2}{k^2 d^2 + s^2 \pi^2}$, the growth rate increases with the increase in medium permeability.

Equation (24) also yields

$$\frac{dn_0}{d\nu} = -\frac{n_0\nu}{2n_0k_1+\nu\varepsilon}, \qquad ...(26)$$

which is always negative. The growth rate of unstable modes, therefore, decreases with the increase in kinematic viscosity.

ACKNOWLEDGEMENTS

It is a pleasure to thank Professor T. J. T. Spanos for helpful discussions while the first author was at the Department of Physics, University of Alberta, where the work was initiated.

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Jñānābha, Vol, 20, 1990

Certain Reducible Cases of Triple Hypergeometric Functions

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(Received : November 28, 1988, Revised : October 4, 1989)

Abstract

Under specialized parameters and arguments, hypergeometric functions of three variables can be reduced to hypergeometric functions of two and one variable, and are expressible into simpler functions notably in terms of elementary functions. This could be achieved by connecting parameters/arguments by one or several relations. In the present paper we consider various hypergeometric functions of three variables and obtain their interesting reducible cases. The importance of the results obtained lies in the fact that they have been derived on using very elementary transformations of Gauss functions.

1. Introduction

The various triple hypergeometric functions, a systematic and extensive documentation of which is given in ([8], Chapters 1 and 3), have been defined and studied by G. Lauricella, S. Saran, H. M. Srivastava, H. Fxton and many others. In this paper we have established numerous reducible cases associated with the well known triple hypergeometric functions F_K, F_N, F_P F_S, F_T, X_{10}, X_{14} and X_{17}. The first five functions (with a set of other five funcitons viz. F_E, F_F, F_G, F_M, F_R), though introduced by Lauricella in 1893, fell into oblivion until 1954 when Saran [5] initiated their systematic study and redefined them in convenient notations. The latter functions X_{10}, X_{14} and X_{17} are among 20 triple hypergeometric functions defined and studied by Exton [4]. For definitions, notations and convergence conditions of these functions, we follow [8].

2. Reduction Formulae

We establish the following reduction formulae :

(2.1) $$F_T\,(\alpha,\ \delta-\alpha,\ \delta-\alpha,\ \delta-\beta,\ \beta,\ \delta-\beta;\ \delta,\ \delta,\ \delta;\ x,\ y,z)$$

$$=(1-x)^{-\alpha}(1-y)^{-\beta}(1-z)^{\alpha+\beta+\delta}\,{}_2F_1\left(\begin{matrix}\alpha,\beta;\\ \delta;\end{matrix}\frac{xy-x-y+z}{1-x-y+xy}\right)$$

(2·2) $$F_K(\alpha,\delta,\delta,\delta,\beta,\delta;\delta,\delta,\delta;x,y,z)$$
$$=(1-z)^{\alpha+\beta-\delta}(1-x-z)^{-\alpha}(1-y-z)^{-\beta}$$
$${}_2F_1\left(\begin{matrix}\alpha,\beta;\\ \delta;\end{matrix}\frac{xyz}{(1-x-z)(1-y-y)}\right)$$

(2·3) $$F_N\left(\alpha,\alpha-\beta,\delta'-\beta,\beta,\beta',\beta;\delta,\delta,\delta;x\frac{y}{y-1},y\right)$$
$$=(1-y)^{-\beta}F_2\left(\alpha,\beta,\beta',\delta,\delta';\frac{x}{1-y},\frac{-y}{1-y}\right),$$

(2·4) $$F_P\left(1+\alpha,\beta,1+\alpha,\alpha,\alpha,\delta;\alpha,\beta+\delta,\beta+\delta;x,\frac{y}{y-1},y\right)$$
$$=(1-y)^{-1-\alpha}H_4\left(1+2\alpha,\beta,\tfrac{1}{2}+\alpha,\beta+\delta;\frac{x}{4(1-y)},\frac{-y}{(1-y)}\right),$$

(2·5) $$F_S\left(\sigma,\alpha,\alpha,\delta-\alpha,\beta,\delta-\beta-\sigma;\delta,\delta,\delta;x,y,\frac{x}{x-1}\right)$$
$$=(1-x)^{\alpha-\sigma}\,{}_2F_1\left(\begin{matrix}\alpha,\beta;\\ \delta;\end{matrix}\;x+y-xy\right),$$

(2·6) $$X_{10}(\alpha,\beta;1+\alpha,\delta;x,y,z)$$
$$=2^{\alpha}[1+\surd(1-4x)]^{-\alpha}H_4\left(\beta,\alpha;\delta,1+\alpha;z,\frac{2y}{1+\surd(1-4x)}\right),$$

(2·7) $$X_{14}(\alpha,\beta,\delta;1+\alpha,\sigma;x,y,z)$$
$$=2^{\alpha}[1+\surd(1-4x)]^{-\alpha}F_2\left(\beta,\alpha,\delta;1+\alpha,\sigma;\frac{2y}{1+\surd(1-4x)}\right),$$

(2·8) $$X_{17}(\alpha,\delta,\beta;\tfrac{1}{2},\delta,\delta;x^2,y,z)$$
$$=\tfrac{1}{2}(1-z)^{-\beta}\left[(1+2x-y)^{-\alpha}\,{}_2F_1\left(\begin{matrix}\alpha,\beta;\\ \delta;\end{matrix}\frac{yz}{(1-z(1+2x-y)}\right)\right.$$
$$\left.+(1-2x-y)^{-\alpha}\,{}_2F_1\left(\begin{matrix}\alpha,\beta;\\ \delta;\end{matrix}\frac{yz}{(1-z)(1-2x-y)}\right)\right].$$

3. Proof of Reduction Formula (2·1)

For convenience, let Ω denote the first member of the reduction for-mula (2·1). Expressing F_T as a triple series, we have

(3·1) $$\Omega=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_m(\delta-\alpha)_{n+p}(\delta-\beta)_{m+p}(\beta)_n\,x^m y^n z^p}{(\delta)_{m+n+p}\,m!\,n!\,p!}$$

$$= \sum_{m,\, p=0}^{\infty} \frac{(a)_m\, (\delta-a)_p\, (\delta-\beta)_{m+p}\, x^m\, z^p}{(\delta)_{m+p}\, m!\, p!}\; {}_1F_1\left(\begin{matrix}\delta-a+p\,\beta; \\ \delta+mp;\end{matrix}\, y\right).$$

Using the following transformation ([3], *p*. 105)

$$(3{\cdot}2) \qquad {}_2F_1\left(\begin{matrix}a,\ \beta; \\ \delta;\end{matrix}\ x\right) = (1-x)^{-\alpha}\ {}_2F_1\left(\begin{matrix}a,\ \delta-\beta; \\ \delta;\end{matrix}\ \frac{-x}{1-x}\right)$$

we obtain

$$(3.3) \qquad \mathfrak{S} = \sum_{m,\, n,\, p=0}^{\infty} \frac{(a)_m\, (\delta-a)_{n+p}\, (\delta-\beta)_{m+n+p}\, x^m\, (-y)^n\, z^p}{(\delta_{.\ m+n+p}\, m!\, n!\, p!\, (1-y)^{n+p+\delta-\alpha}}$$

$$=(1-y)^{\alpha+\delta} \sum_{n,\, p=0}^{\infty} \frac{(\delta-\alpha)_{n+p}\, (\delta-\beta)_{n+p}\, (-y)^n\, z^p}{(\delta)_{n+p}\, n!\, p!\, (1-y)^{n+p}}\, {}_2F_1\left(\begin{matrix}\delta-\beta+n+p; \\ \delta+n+p;\end{matrix}\, x\right)$$

again using the transformation (3.2) we obtain.

$$(3{\cdot}4) \qquad \mathfrak{S} = (1-y)^{\alpha-\delta}\ (1-x)^{\beta-s} \sum_{m,\, n,\, p=0}^{\infty} \frac{(\delta-a)_{m+n+p}\, (\delta-\beta)_{m+n+p}}{(\delta)_{m+n+p}\, m!\, n!\, p!}$$

$$[-x/(1-x)]^n\, [-y/(1-x)\,(1-y)]^n\, [z/(1-x)\,(1-y)]^p.$$

The triple series can be reduced to ordinary sries on using the following result (see, for example [7], *p*. 19)

$$(3{\cdot}5) \qquad \sum_{m,\, n,\, p=0}^{\infty} f(m+n+p)\, \frac{x^m\, y^n\, z^p}{m!\, n!\, p!} = \sum_{n=0}^{\infty} f(n)\, \frac{(x+y+z)^n}{n!}$$

and thus we obtain from (3.4)

$$(3{\cdot}6) \qquad \mathfrak{S} = (1-x)^{\beta-\delta}\, (1-y)^{a-\delta}\ {}_2F_1\left(\begin{matrix}\delta-a,\ \delta-\beta; \\ \delta;\end{matrix}\ \frac{xy-x-y+z}{(1-x)(1-y)}\right),$$

which, in view of the transformation ([2], Sec 1.2 (2)), is precisely the second member of (2.1) and the proof of (2.1) is thus completed. This result is also deducible from ([6], (5.19) or (5.20)) by reducing F_1 or F_3 to Gauss function by means of reduction formulae ([7], 9.4(99), 9.4(87); 9.4 (79), (80), (81)).

The proofs of other results are simple; the function is first expressed in terms of a series involving hypergometric series and the known results are applied. We then simplify and interpret the resulting expression as hypergeometric series to obtain the results in their final form. For example, to prove the reduction formula (2.2), we make use of the result ([3], *p*. 238, eqn. 3). The application of the result ([8], *p.* 302, eqn. 92) leads to the formulae (2.3), (2.4) and (2.5). The result ([3], *p.* 101, eqn 6) is applied to prove the reduction formula (2.6) and (2.7). While deriving the result (2.8) we have used the result ([3], *p.* 101, eqn. 5).

We conclude by observing that several interesting special cases of our reduction formulae can be obtianed on suitably specializing the parameters.

Acknowledgements

The authors are thankful to Prof. H. M. Srivastava, Department of Mathematics, University of Victoria, Canada, for his fruitful suggestions.

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Jñānābha, Vol. 20, 1990

Some m-Dimensional Generating Relations for $\mathbf{B}_n^A\{(x_m), y\}$

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(Received : February 2, 1989, Revised : July 21, 1989)

ABSTRACT

Recently, we introduced a general polynomial svstem $B_n^A\ \{(x_m, y\}$ of $m+1$ variables with the help of a generating relation involving the generalized hypergeometric ${}_pF_q$ and the Lauricella function F_A. This paper enunciates a linear and a bilateral generating relation for this polynomial system. Fiaally, several multilinear and multilateral generating relations have been deduced as special cases of our results.

1. INTRODUCTION

A general polynomial system$\left\{ B_n^A\ \{(x_m), y\} \,\middle|\, n=0, 1, 2, \ldots \right\}$ has been defined by us as follows [1] :

$$\sum_{n=0}^{\infty} B_{n;\, \nu,\, (\gamma m;\, 1),\, (c_m),\, (\beta_q)}^{A;\, m_1,\, m_2,\, a,\, (b_m),\, (a_p),\, (\mu_m)} \{(x_m), y\} t^n$$

$$= {}_pF_q\left[(a_p);\, (\beta_q);\, \nu y^{m_2}\ \ t^{m_1} \right] F_A\left[a, (b_m); (c_m); \mu_1 x_1 t, \right.$$

$$\left. \mu_2 x_2 t^{\gamma_2}, \ldots, \mu_m x_m t^{\gamma_m} \right]; \qquad \ldots(1{\cdot}1)$$

where ${}_pF_q$ is the generalized hypergeometric function and F_A is one of the Lauricella functions [2, *p*. 41].

The conditions for validity of (1·1), and the notations and abbreviations used here, are alreeady given in our earlier work [1]. Furthermore, we may recall an explicit representation for the polynomial $B_n^A\{(x_m),\ y\}$ in the form (cf. [1]) :

$$B_n^A\{(x_m),\ y\} = \sum_{r_1=0}^{[n/m_1]}\ \sum_{r_2=0}^{[(n-m_1)/\gamma_2]} \cdots \sum_{r_m=0}^{\left[(n-m_1 r_1-\sum_{i=2}^{m-1}\gamma_i r_i)/\gamma_m\right]}$$

$$\frac{(a)_{n-m_1r_1-\sum_{i=2}^{m}(\gamma_i-1)\gamma_i}\ (b_1)_{n-m_1r_1-\sum_{i=2}^{m}\gamma_i r_i}\ ((\alpha_p)_{r_1})}{(n-m_1 r_1-\sum_{i=1}^{m}\gamma_i r_i)!\ (c_1)_{n-m_1 r_1-\sum_{i=2}^{m}\gamma_i r_i}\ ((\beta_q)_{r_1})}$$

$$\cdot \frac{\left[(b_{m;1})_{r_{m;1}}\right](\nu y^{m_2})^{r_1}\ (\mu_1 x_1)^{n-m_1r_1-\sum_{=2}^{m}\gamma_i r_i}\ \left[(\mu_{m;1}\, x_{m;1})^{r_{m;1}}\right]}{\left[(c_{m;1})_{r_{m;1}}\right]\ \left[(r_m!)\right]} \qquad \ldots(1.2)$$

In this paper, we derive a linear and a bilateral general generating relation, which yield, as their applications, various multidimensional generating relations.

2. LINEAR GENERATING RELATION

$$\sum_{n=0}^{\infty} \frac{\left[(\delta_N)_n\right]\left[(\delta'_{N'})_n\right]}{\left[(\eta_M)_n\right]} B_n^A\{(x_n),\ y\}\ t^n =$$

$$= F^{N+N'+1\,:\,1,\,p;\,1,\,\ldots,\,1}_{M\,:\,q;\,1,\,\ldots,\,1}\left[\begin{array}{l}[(\delta_N):1,\, m_1,\, \gamma_2,\, \ldots,\, \gamma_m],\ [(\delta'_{N'}):1,\, m_1,\, \gamma_2,\ldots,\, \gamma_m] \\ [(\eta_M):1,\, m_1,\, \gamma_2,\, \ldots,\, \gamma_m],\ -,\end{array}\right.$$

$$(a:1,\,0,\,1,\,\ldots,\,1):(b_1;\,1),\,(a_1;\,1),\,\ldots,\,(a_p;\,1);\,(b_2;\,1);\,\ldots,\,(b_m;\,1);$$

$$-\ :(c_1;\,1),\,(\beta_1;\,1),\,\ldots,\,(\beta_q;\,1);\,(c_2;\,1);\,\ldots;\,(c_m;\,1);$$

$$\left.\nu y^{m_2}\, t,\ \mu_1 x_1\, t^{m_1},\ \mu_2\, x_2\, t^{\gamma_2},\, \ldots,\, \mu_m x_m t^{\gamma_m}\right], \qquad \ldots(2.1)$$

which may also be written as

$$\sum_{n=0}^{\infty} \frac{[(\delta_N)_n]\,[(\delta'_{N'})_n]}{[(\eta_M)_n]}\ B_n^A\{(x_m),\ y\}\ t^n$$

$$
=\sum_{n=0}^{\infty}\sum_{r_1=0}^{\infty}\cdots\sum_{r_{m-1}=0}^{\infty}\frac{\left[(\delta_N)_{n+m_1 r_1+\sum_{i=2}^{m-1}\gamma_i r_i}\right]}{\left[(\eta_M)_{n+m_1 r_1+\sum_{i=2}^{m-1}\gamma_i r_i}\right]}
$$

$$
\cdot\frac{\left[(\delta'_{N'})_{n+m_1 r_1+\sum_{i=2}^{m-1}\gamma_i r_i}\right](a)_{n+r_2+\ldots+r_{m-1}}(b_1)_n\,((ap)_{r_1})}{n!\qquad (c_1)_n\qquad ((\beta q)_{r_1})}
$$

$$
\frac{\left[(b_{m-1;\,1})_{r_{m-1;\,1}}\right](\nu y^{m_2})^{r_1}(\mu_1 x_1)^n\left[(\mu_{m-1;\,1}\,x_{m-1;\,1})^{r_{m-1;\,1}}\right]}{\left[(c_{m;\,1;\,1})_{r_{m-1;\,1}}\right]\qquad\left[(r_{m-1}!)\right]}
$$

$$
{}_{(N+N')\gamma_{m+2}}F_{M\gamma_{m+1}}\left[\begin{matrix}\Delta(\gamma_m;(\delta_N)+n+m_1 r_1+\sum_{i=2}^{m-1}\gamma_i r_i),\\ \\ \Delta(\mu_m;(\eta_M)+n+m_1 r_1+\sum_{i=2}^{m-1}\gamma_i r_i)\end{matrix}\right.
$$

$$
\left.\begin{matrix}\Delta(\gamma_m;(\delta'_{N'})+n+m_1 r_1+\sum_{i=2}^{m-1}\gamma_i r_i),\ a+n+r_2+\ldots+\gamma_{m-1},\ b_m; & \\ & K_m x_m t^{\gamma_m}\\ -,\qquad\qquad -,\qquad\qquad -,\ c_m; & \end{matrix}\right] \quad \ldots(2.2)
$$

where

$$
K_m=\gamma_m^{(N+N'-M)\gamma_m}.
$$

3. BILATERAL GENERATING RELATION

With the help of the generalized Lauricella function of Srivastava and Daoust (cf. [2] and [4]), and using (2.2), we obtain the following bilateral generating relation for the polynomial system (1.1):

$$\sum_{n=0}^{\infty} \frac{[(\delta_M)_n]\,[\delta'_M)_n]}{[(\eta_N)_n]}\; B_n^A\,\{(x_m),y\}\; F^{A':\,B'+2,\,\ldots,\,B^{(M)}+2}_{C:\,D',\,\ldots,\,D^{(M)}} \left[\begin{matrix}[(\psi);1,\ldots,1]: & (\delta_1+n;1),(\delta'+n;1),[(b');1];\ldots;(\delta_M+n;1),\\ [(\gamma);1,\ldots,1]: & [(d');1]:\ldots;\\ & (\delta'_M+n;1),[(b)^{(M)};1]; \\ & [(d^{(M)});1];\end{matrix}\; Z_1,\ldots,Z_M\right] t^n$$

$$=\sum_{n=0}^{\infty}\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty} \frac{\left[{}^{(\delta_M)}\,n+m_1r_1+\sum_{i=2}^{m}\gamma_i r_i\right]\left[\left[{}^{(\delta'_M)}\,n+m_1r_1+\sum_{i=2}^{m}\gamma_i r_i\right]\right.}{\left[{}^{(\eta_N)}\,n+m_1r_1+\sum_{i=2}^{m}\gamma_i r_i\right]}$$

$$\frac{(a)_{n+r_2+\ldots+r_m}(b_1)_n\left((\alpha_p)_{r_1}\right)\left[(b_{m;1})_{r_m;1}\right](\nu y^{m_2}\;t^{m_1})^{r_1}}{(c_1)_n\left((\beta_q)_{r_1}\right)\left[(c_{m;1})_{r_m;1}\right]}$$

$$\frac{(\mu_1\,x_1\,t)^n\left[(\mu_{m;1}\,x_{m;1}\,t^{\gamma_{m;1}})^{r_{m;1}}\right]}{[(r_m!)]}$$

$$F^{A':\,B'+2,\,\ldots,\,B^{(M)}+2}_{C:\,D',\,\ldots,\,D^{(M)}}\left[\begin{matrix}[(\psi);1,\ldots,1]: & (\delta_1+n+m_1r_1+\sum_{i=2}^{m}\gamma_i r_i;1)\\ [(\gamma);1,\ldots,1]: & [(d');1],\\ & (\delta'_1+n+m_1r_1+\sum_{i=2}^{m}\gamma_i r_i:1),[(b'),1]\ldots;(\delta_M+n+m_1r_1+\sum_{i=2}^{m}\gamma_i r_i;1),\\ & [(d^{(M)});1]\\ & (\delta'_M+n+m_1r_1+\sum_{i=2}^{m}\gamma_i r_i:1),[(b^{(M)});1];\\ & -;\end{matrix}\; Z_1\ldots,Z_M\right] \tag{3.1}$$

4. APPLICATIONS

(*i*) Substituting $p=q$, $(\alpha_p)=(\beta_q)$, $\nu=0$, $m=2=\gamma_2=\mu_1$, $(b_2)=(c_2)$, $a=\frac{1}{2}$, $x_1=x$, $\mu_2=-1$, $x_2=1$, $\delta_i=1=\delta'_i=\eta_i$, $z_j=\dfrac{1-Z_j}{1+Z_j}$, $(j=1,2,\ldots,M$, $B'=\ldots=B^{(M)}=0$, $D'=\ldots=D^{(M)}=1$, $A'=C$, $\psi=\gamma$, relation (3.1) yields :

$$\sum_{n=0}^{\infty}(n!)^M P_n(x)\;P_n(Z_1)\ldots P_n(Z_M)\left\{\frac{2}{(1+Z_1)}\cdots\frac{2}{(1+Z_M)}\right\}^{-nM+1} t^n$$

$$= \sum_{n,\, m_1=0}^{\infty} \frac{\left\{(1)_{n+2m_1}\right\}^M (1/2)_{n+M_1} (2xt)^n (-t^2)^{m_1}}{n!\, m_1!}$$

$$\cdot \prod_{j=1}^{M} {}_2F \left[\begin{matrix} 1+n+2m_1,\ 1+n+2m_1\,; \\ 1\,; \end{matrix} \frac{1-Z_j}{1+Z_j} \right], \tag{4.1}$$

which is a multilinear generating relation for Legendre polynomials.

(11) Under suitable conditions, (3·1) gives

$$\sum_{n=0}^{\infty} \{(\gamma+1/2)_n\}^{M-1} \{2\gamma)_n\}^{M-1} \left\{ \frac{n!}{(2\gamma)_n} \right\}^M \left\{ \left(\frac{1+Z_1}{2} \right)^n \cdot\cdot \left(\frac{1+Z_M}{2} \right)^n \right\}$$

$$\cdot\ L_n^{\alpha}(x)\ C_n^{\gamma}(Z_1)\ \ldots\ C_n^{\gamma}(Z_M)\ t^n$$

$$= \sum_{n,\, r_1=0}^{\infty} \frac{\{(\gamma+\tfrac{1}{2})_{n+r_1}\}^{M-1} (2xt)^n\ (-t^2)^{r_1}}{n!\ r_1!\ (1+\alpha)_n} \{(2\gamma)_{n+r_1}\}^{M-1}$$

$$\cdot\ (1+\alpha)_{n+r_1} \left\{ \left(\frac{1+Z_1}{2} \right) \cdots \left(\frac{1+Z_M}{2} \right) \right\}^{n+r_1} \left\{ \frac{(n+r_1)!}{(2\gamma)_{n+r_1}} \right\}, \qquad \ldots(4.2)$$

where $C_n^{\gamma}(Z)$ is the Gegenbauer polynomials [3, *p*. 277].

This is believed to be a new generating relation. In fact, for $M=1$, $t=2T/(1-Z_1)$, and $1+\alpha=2\gamma$, it gives [3, *p*. 281, Eqn. (24)]

$$\sum_{n=0}^{\infty} \frac{n!}{(2\gamma)_n}\ C_n^{\gamma}(Z_1)\ L_n^{(2\gamma-1)}(x)\ T^n = \rho^{-2\gamma} \exp\left[\frac{-xT(Z_1-T)}{\rho^2} \right]$$

$$\cdot\ {}_0F_1 \left[-\,;\ \gamma+\tfrac{1}{2};\ \frac{x^2T^2(Z_1^2-1)}{4\rho^2} \right], \qquad \ldots(4.3)$$

$$\rho = (1-2xT+T^2)^{1/2}.$$

(*iii*) Applying suitable conditions on (3.1), we have

$$\sum_{n=0}^{\infty} \frac{\{(1+2\alpha)_n\}^{M+1}\ \{(1+\alpha)_n\}^{M-1}}{(n!)^M} \left\{ \left(\frac{2}{1+Z_1} \right) \cdots \left(\frac{2}{1+Z_M} \right) \right\}^{-(1+2\alpha+n)}$$

$$P_n^{(\alpha,\,\alpha)}(x)\, P_n^{(\alpha,\,\alpha)}(Z_1) \ldots P_n^{(\alpha,\,\alpha)}(Z_M)\ t^n$$

$$= \sum_{n,\, r_1=0}^{\infty} \frac{\{(1+2\alpha)_{n+r_1}\}^M\ \{(1+\alpha)_n\}^{M-1}\ (2xt)^n\ (-t^2)^{r_1}}{n!\ r_1!\ (1)_{n+2r_1}}$$

$$\cdot \prod_{i=1}^{M} {}_2F_1\left[\begin{matrix} 1+2\alpha+n+2r_1,\ 1+\alpha+n+2r_1 \\ 1+\alpha \end{matrix} ;\ \frac{1-Z_i}{1+Z_i}\right], \qquad \ldots(4.4)$$

which is a multilinear generating relation for Gegenbauer polynomials involved in (4.2) and (4.3).

Various other relations can similarly be deduced. Several known bilinear, bilateral, trilinear, and trilateral generating relations happen to be special cases of our results (4.1), (4.2) and (4.4).

ACKNOWLEDGEMENTS

The authors express their sincere thanks to Professor H. M. Srivastava (Victoria, Canada) for his valuable suggestions.

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Jñānābha; Vol. 20, 1990

Some Finite Double Integral Formulas Involving Fox's H-Function

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(Recived : July 26, 1989, Revised : November 29, 1989)

ABSTRACT

In the present paper we evaluate three finite double integrals involving various products of biorthogonal pair of polynomials, a general class of polynomials, and Fox's H-function with general arguments. Our integrals are quite general in character and a number of (known or new) integrals can be deduced as particular cases. Some interesting special cases of our main results have been discussed briefly.

1. INTRODUCTION AND DEFINITION

The biorthogonal pair of polynomial sets occurring in this paper are defined and represented in the following manner [1] :

$$J_n^{(\alpha,\beta)}(x;k)=\frac{(\alpha+1)_{kn}}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(\alpha+\beta+n+1)_{kj}}{(\alpha+1)_{kj}}\left(\frac{1-x}{2}\right)^{kj} \qquad \text{...(1.1)}$$

and

$$K_n^{(\alpha,\beta)}(x;k)=\frac{1}{n!}\sum_{j=0}^{n}(-1)^j\binom{\beta+n}{j}\left(\frac{x-1}{2}\right)^j\left(\frac{x+1}{2}\right)^{n-j}$$

$$\cdot\sum_{w=0}^{j}(-1)^w\binom{j}{w}\left(\frac{\alpha+w+1}{k}\right)_n \qquad \text{...(1.2)}$$

$$(\mathrm{Re}\,(\alpha)>-1,\ \mathrm{Re}(\beta)>-1).$$

When $k=1$, both $J_n^{(\alpha,\beta)}(x;k)$ and $K_n^{(\alpha,\beta)}(x;k)$ reduce to the classical Jacobi polynomials (see also Srivastava and Manocha [7, P. 197, Problem 64]).

Srivastava [5, p. 1, Eq. (1)] introduced the general class of polynomials (see also Srivastava and Singh [8]) as

$$S_n^m(x)=\sum_{u=0}^{[n/m]}\frac{(-n)_{mu}}{u!}A_{n,u}X^u\ (n=0,1,2,\ldots), \qquad \text{...(1.3)}$$

where m is arbitrary positive integer, and the coefficients $A_{n,\,u}$ $(n, u \geqslant 0)$ are arbitrary constants, real or complex.

2. MAIN INTEGRALS

We shall evaluate the following three general double integrals :

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\, y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}\ {}_2F_1\left[\begin{matrix}a, b; \\ 1+a-b;\end{matrix}\ \frac{(1+t)y}{1+ty}\right]$$

$$\cdot\ J_s^{(\alpha,\,\beta)}(1-2x;k)\ S_n^m\left[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}\right]$$

$$\cdot\ H_{PQ}^{MN}\left[zx^{\sigma}(I-x)^{\nu}\left\{\frac{y(1+ty)}{(1-y)}\right\}^{T}\,\middle|\,\begin{matrix}(a_j, \alpha_j)_{1,\,P} \\ (b_j, \beta_j)_{1,\,Q}\end{matrix}\right]dx\,dy$$

$$=\frac{2^{a}\{4(1+t)\}^{-\rho}\,\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\,(\alpha+1)_{ks}}{\sqrt{\pi}\,\Gamma(1+a\ \Gamma\left(1+\frac{a}{2}-b\right)\quad s!}\sum_{u=0}^{[n/m]}\sum_{j=0}^{s}\frac{(-n)_{mu}}{u!}$$

$$\cdot\ \frac{(\alpha+\beta+s+1)_{kj}(-s)_j}{(\alpha+1)_{kj}\,j!\,\{4(1+t)\}^{\gamma u}}\,A_{n,u}\,H_{P+4,\ Q+3}^{M+2,\ N+3}\left[z\{4(1+t)\}^{-T}\,\middle|\,\begin{matrix}A \\ B\end{matrix}\right] \qquad \text{...(2.1)}$$

where

$A=(1-\mu, \nu), (1-\lambda--kj, \sigma), (1-\rho-\gamma u, T), (a_j, \alpha_j)_{1,\,P}\ (1+a-b-\rho-\gamma u, T)$

$B=\left(\frac{1}{2}+\frac{a}{2}-\rho-\gamma u, T\right), \left(1+\frac{a}{2}-b-\rho-\gamma u, T\right) (b_j, \beta_j)_{1,\,Q}$

$(1-\lambda-\mu-kj, \sigma+\nu)$

The integral (2.1) is valid if the following sets of (sufficient) conditions are satisfied.

(*i*) m is arbitrary positive integer, and the coefficient $A_{n,\,u}$ $(n, u \geqslant 0)$ are arbitrary constants, real or complex.

(*ii*) $\Omega = \sum_{j=1}^{N}\alpha_j - \sum_{j=N+1}^{P}\alpha_j + \sum_{j=1}^{M}\beta_j - \sum_{j=M+1}^{Q}\beta_j > 0,$

and $|\arg z| < \frac{1}{2}\Omega\pi$

(*iii*) $\text{Re}(1+a-b)>0$, $\text{Re}(\alpha)>-1$, $\text{Re}(\beta)>-1$, $t>-1$ $\gamma\geqslant 0$, $\text{Re}(\lambda+\sigma\xi)>0$, $\text{Re}(u+\nu\xi)>0$, $\text{Re}(\rho+\gamma u+T\xi)>0$, $\text{Re}(1+a-2\rho-2\gamma u-2T\xi)>0$. where $\xi=\min\limits_{1\leqslant j\leqslant m}[\text{Re}(b_j/\beta_j)]$ and $u=0, 1, 2, \ldots [n/m]$.

(*iv*) $\text{Re}(1-2b)>0$.

Second Integral

$$\int_0^1 \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} y^{\rho-1} (1+ty)^{\rho-a-1} (1-y)^{a-2\rho}$$

$$ {}_3F_2 \left[\begin{matrix} a, \frac{a}{2}+1, b \\ \frac{a}{2}, 1+a-b \end{matrix} ; \frac{(1+t)y}{1+ty} \right]$$

$$\cdot J_s^{(\alpha, \beta)} (1-2x, k)\, S_n^m [cy^\gamma (1+ty)^\gamma (1-y)^{-2\gamma}]$$

$$\cdot H_{P, Q}^{M, N} \left[z\, x^\sigma (1-x)^\nu \left\{ \frac{y\,(1+ty)}{(1-y)^2} \right\}^T \middle| \begin{matrix} (a_j, \alpha_j)_{1,\,P} \\ (b_j, \beta_j)_{1,\,Q} \end{matrix} \right] dx\, dy$$

$$= \frac{2^a \{4\,(1+t)\}^{-\rho}\, \Gamma(1+a-b)\, (\alpha+1)_{ks}}{\sqrt{\pi}\;\; \Gamma(1+a)\;\; s!} \sum_{u=0}^{[n/m]} \sum_{j=0}^{s} \frac{(1-n)_{mu}}{u!}$$

$$\cdot \frac{(\alpha+\beta+s+1)_{kj}\, (-s)_j\, c^u}{(\alpha+1)_{kj}\; j!\, \{4\,(1+t)\}^{u\gamma}}\, A_{n,\,u}\; H_{P+4,\, Q+3}^{M+2,\, N+3} \left[z\, \{4\,(1+t)\}^{-T} \middle| \begin{matrix} A \\ C \end{matrix} \right] \qquad \text{...(2.2)}$$

where

A is given with the integral (2.1), and

$$C = \left(\frac{1}{2} + \frac{a}{2} - \rho - \gamma u, T \right), \left(1 + \frac{a}{2} - \rho - \gamma u, T \right) (b_j, \beta_j)_{1,\,Q},$$

$$(1-\lambda-\mu-kj, \sigma+\nu).$$

Also, the conditions of validity of (2.2) are the same as those mentioned with (2.1), except that here we have $0 \leqslant b < 1$ instead of $Re\,(1-2b) > 0$.

Third Integral

$$\int_0^1 \int_0^1 x^{\lambda-1} (1-x)^{\mu-1}\, y^{\rho-1} (1+ty)^{\rho-a-1} (1-y)^{a-2\rho}$$

$$ {}_4F_3 \left[\begin{matrix} a, \frac{a}{2}+1, c, d; \\ \frac{a}{2}, 1+a-c, 1+a-d; \end{matrix} \frac{(1+t)\,u}{1+tu} \right]$$

$$\cdot J_s^{(\alpha, \beta)} (1-2x; k)\, S_n^m [cy^\gamma (1+ty)^\gamma (1-y)^{-2\gamma}]$$

$$\cdot H_{P, Q}^{M, N} \left[zx^\sigma (1-x)^\nu \left\{ \frac{y\,(1+ty)}{(1-y)^2} \right\}^T \middle| \begin{matrix} (a_j, \alpha_j)_{1,\,P} \\ (b_j, \beta_j)_{1,\,Q} \end{matrix} \right] dx\, dy$$

$$= \frac{2^a \{4\,(1+t)\}^{-\rho}\, \Gamma(1+a-c)\, \Gamma(1+a-d)\, (\alpha+1)_{ks}}{\sqrt{\pi}\;\; \Gamma(1+a)\, \Gamma(1+a-c-d)\;\; s!} \sum_{u=0}^{[m/n]} \sum_{j=0}^{s} \frac{(-n)_{mu}}{u!}$$

$$\cdot \quad \frac{(a+\beta+s+1)_{kj}\,(-r)_j\, c^u}{(a+1)_{kj}\, j!\, \{4\,(1+t)\}^{u\gamma}} A_{n,\,u}\, H^{M+3,\, N+3}_{P+5,\, Q+4}\left[z\,\{4\,(1+t)\}^{-T} \,\middle|\, \begin{matrix} D \\ E \end{matrix} \right] \qquad \ldots(2.3)$$

where

$$D=(1-\mu,\, \nu),\ (1-\lambda-kj,\, \sigma),\ (1-\rho-\gamma u,\, T),\ (a_j,\, \alpha_j)_{1,\,P},\ (1+a-c-\rho-\gamma u,\, T),$$
$$(1+a-d-\rho-\gamma u,\, T)$$

$$E=\left(\frac{1}{2}+\frac{a}{2}-\rho-\gamma u,\, T\right),\ \left(1+\frac{a}{2}-\rho-\gamma u,\, T\right),\ (1+a-c-d-\rho-\gamma u,\, T),$$
$$(b_j,\, \beta_j)_{1,\,Q},\ (1-\lambda-\mu-kj,\, \sigma+\nu).$$

The integral (2.3) is valid if the following set of (sufficient) conditions are satisfied.

(1) Re $(1+a-c)>0$, Re $(1+a-2c-2d)>0$. Re $(1+a-d)>0$.

(2) The sets of conditions (*i*) to (*iii*) mentioned with (2.1) also hold.

Remark. We can also evaluate three more integrals by replacing $J_x^{(a,\,\beta)}$ $(x;\, k)$ by $K_n^{(a,\,\beta)}$ $(x;\, k)$, but we prefer to omit the details. Throughout this paper, $H^{M,\,N}_{P,\,Q}\,[x]$ stands for the well-known Fox's H-function. (The details this function can be found, for example, in Srivastava, Gupta and Goyal [6]. Here $(a_j,\, \alpha_j)_{1,\,p}$ abbreviates the parameter sequence $(a_1,\, \alpha_1)\, \ldots\, (a_p,\, \alpha_p)$

Proofs. To establish (2.1), we use the series representation of $S_n^m\,(x)$ as given by (1.2) and for the *H*-function we use its Mellin-Barnes types of contour integral (6, p. 10, Eq. (2.1.1.)], on the left-hand side of (2.1), and change the order of integration summation (which is justified under the conditions given with (2.1) . We then obtain

$$\text{Left-hand side of } (2.1)= \sum_{u=0}^{[n/m]} \frac{(-n)_{mu}}{u!}\, c^u\, A_{n,\,u} \int_{\alpha} \Bigg[\phi\,(\xi)\, z^{\xi}$$

$$\cdot \int_0^1 x^{\lambda+\sigma\xi-1}\,(1-x)^{\mu+\nu\xi-1}\, J_s^{(a,\,\beta)}\,(1-2x;\, k)\, dx$$

$$\cdot \int_0^1 y^{\rho+\gamma u+T\xi-1}\,(1-y)^{a-2T\xi-2\gamma u-2\rho}\,(1+ty)^{\rho-a+\gamma u+T-1}.$$

$$\cdot\ {}_2F_1\left[\begin{matrix} a,\, b; \\ 1+a-b; \end{matrix}\ \frac{(1-t)y}{1+ty}\right] dy \Bigg]\, d\xi$$

Now using known results [2, *p*. 118, Eq. (3.3.1)] and [3, *p*. 254, Eq. (2.1)], and interpreting the resulting contour integral as the *H*-function, we at once get the right-hand side of (2.1).

The proofs of the second and third integrals are similar to that of the first integral with the only difference that here we make use of known integrals [3, *p.* 254, *Eq.* (2.2)] and [3, *p.* 255, *Eq.* (2.3)], respectively, instead of [3. *p.* 254, *Eq.* (2.1)].

3. SPECIAL CASES

(*i*) If we put $m=1$ and $A_{n,\,u}=\dfrac{(\alpha'+1)_n\,(\alpha'+\beta'+n+1)_u}{n!(\beta'+1)_u}$ in (2.1), then $S_n^1(x)\rightarrow P_n^{(\alpha',\,\beta')}(1-2x)$, consequently we arrive at the following interesting integral :

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\,y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}\,{}_2F_1\left[\begin{matrix}a,\,b;\\1+a-b;\end{matrix}\,\frac{(1+t)y}{1+ty}\right]$$

$$\cdot J_n^{(\alpha,\,\beta)}(1+2x;\;k)\;P_n^{(\alpha',\,\beta')}\left[1-2\{cy^{\gamma}(1-y)^{-2\gamma}(1+ty)^{\gamma}\}\right]$$

$$\cdot H_{P,\,Q}^{M,\,N}\left[z\,x^{\sigma}(1-x)^{\nu}\left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{T}\left|\begin{matrix}(a_j,\,\alpha_j)_{1,\,P}\\(b_j,\,\beta_j)_{1,\,Q}\end{matrix}\right.\right]dx\;dy$$

$$=\frac{2^{a}\,\{4(1+t)\}^{-\rho}\,\Gamma(1+a/2)\,\Gamma(1+a-b)_u}{\sqrt{\pi}\,\Gamma(1+a)\,\Gamma(1+a/2-b)}\;\frac{(\alpha+1)_{ks}}{s!}\;\frac{(\alpha+1)_n}{n!}$$

$$\cdot\sum_{u=0}^{n}\;\sum_{u=0}^{s}\frac{(-n)_u}{u!\,j!}\;\frac{(\alpha'+\beta'+n+1)_u}{(\beta'+1)_u}(-1)^j\;\frac{(\alpha+\beta+s+1)_{kj}\;c^u}{(\alpha+1)_{kj}\;\{4(1+t)\}^{\gamma u}}$$

$$\cdot H_{P+4,\,Q+3}^{M+2,\,N+3}\left[z\{4(1+t)\}^{-T}\left|\begin{matrix}A\\B\end{matrix}\right.\right], \qquad \ldots\ 3.1$$

where A and B are given by (2.1), and the sets of conditions (*i*) to (*iv*) mentioned with (2.1) are satisfied.

(*ii*) If we put $k=1$ in (3.1), then we get

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\,y^{\rho-1}(1+ty)^{\rho-a-1}(1-y)^{a-2\rho}$$

$${}_2F_1\left[\begin{matrix}a,\,b;\\1+a-b;\end{matrix}\,\frac{(1+t]y}{1+ty}\right]$$

$$\cdot P_n^{(\alpha,\,\beta)}(1-2x)\;P_n^{(\alpha',\,\beta')}\left[1-2cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}\right].$$

$$\cdot H_{P,\,Q}^{M,\,N}\left[(zx^{\sigma}(1-x)^{\gamma}\left\{\frac{y(1+ty)}{(1+y)^2}\right\}^{T}\left[\begin{matrix}(a_j,\,\alpha_j)_{1,\,P}\\(b_j,\,\beta_j)_{1,\,Q}\end{matrix}\right]\right]dx\;dy$$

$$=\frac{2a\{4(1+t)\}^{-\rho}\,\Gamma(1+a/2)\,\Gamma(1+a-b)\,(\alpha+1)_s\,(\alpha+1)_n}{\sqrt{\pi}\qquad\Gamma(1+a)\;\;\Gamma(1+a/2-b)\;\;s!\quad n!}$$

$$\cdot\sum_{u=0}^{n}\sum_{j=0}^{s}\frac{(-n)_u\,(\alpha'+\beta'+n+1)_u}{(\beta'+1)_u}\frac{(\alpha+\beta+s+1)_j\,c^u}{(\alpha+1)_j}\frac{}{\{4(1+t)\}^{\gamma u}}$$

$$\cdot H_{P+4,\,Q+3}^{M+2,\,N+3}\left[z\{4(1+t)\}^{-T}\left|\begin{matrix}A'\\B'\end{matrix}\right.\right], \qquad \text{...(3.2)}$$

where

$$A'=(1-u,\,\nu),\,(1-\lambda-j,\,\sigma),\,(1-\rho-\gamma u,\,T),\,(a_j,\,\alpha_j)_{1,\,P},\,(1+a-b-\lambda-\gamma u,\,T)$$

$$B'=\left(\frac{1}{2}+\frac{a}{2}-\rho-\gamma u,\,T\right),\,(1+a/2-b-\rho-\gamma u,\,T),\,(b_j,\,\beta_j)_{1,\,Q},$$
$$(1-\lambda-\mu-j,\,\sigma+\nu).$$

(*iii*) Again, let $n=\gamma=0$ in (2.1), $S_n^m(x)$ reduces to $A_{0,\,0}$, and we get the following double integral, which is also believed to be new.

$$\int_0^1\int_0^1 x^{\lambda-1}\,(1-x)^{\mu-1}\,y^{\rho-1}\,(1+ty)^{\rho-a-1}(1-y)^{a-2\rho}\;{}_2F_1\left[\begin{matrix}a,\,b\\1+a-b\end{matrix};\frac{(1+t)y}{1+ty}\right]$$

$$\cdot J_s^{(\alpha,\,\beta)}(1-2x;\,1)\,H_{P,\,Q}^{M,\,N}\left[z\,x^{\sigma}\,(1-x)^{\nu}\left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{T}\left|\begin{matrix}(a_j,\,\alpha_j)_{1,\,P}\\(b_j,\,\beta_j)_{1,\,Q}\end{matrix}\right.\right]dx\,dy$$

$$=\frac{2^a\{4(1+t)\}^{-\rho}\,\Gamma(1+a/2\;\Gamma(1+a-b)}{\sqrt{\pi}\;\Gamma(1+a)\;\Gamma\left(1+\frac{a}{2}-b\right)}\frac{(\alpha+1)_{ks}}{s!}$$

$$\sum_{j=0}^{s}\frac{(\alpha+\beta+s+1)_{kj}\,(-s)}{(\alpha+1)_{kj}\qquad j!}$$

$$\cdot H_{P+4,\,Q+3}^{M+2,\,N+3}\left[z\{4(1+t)\}^{-T}\left|\begin{matrix}A''\\B''\end{matrix}\right.\right], \qquad \text{...(3.3)}$$

where

$$A''=(1-\mu,\,\nu),\,(1-\lambda-kj,\,\sigma),\,(a_j,\,\alpha_j)_{1,\,P},\,(1+a-b-\rho,\,T)$$
$$B''=(\tfrac{1}{2}+a/2-\rho,\,T),\,(1+a/2-b-\rho,\,T),\,(b_j,\,\beta_j)_{1,\,Q},\,(1-\lambda-kj,\,\sigma+\nu).$$

iv) Finally, we let $\sigma,\,\nu\to 0$ in (2.1) and evaluate x-integral in the resulting expression. We arrive at the following single integral

$$\int_0^1 y^{\rho-1}\,(1-y)^{a-2\rho}\,(1+ty)^{\rho-a-1}\;{}_2F_1\left[\begin{matrix}a,\,b\\1+a-b;\end{matrix};\frac{(1+t)y}{1+ty}\right]$$

$$\cdot S_{n}^{m}\left[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}\right] H_{P,\,Q}^{M,\,N}\left[\left\{\frac{y(1+tu)}{(1-y)^{2}}\right\}^{T}\left|\begin{matrix}(a_j,\,a_j)_{1,\,P}\\ (b_j,\,\beta_j)_{1,\,Q}\end{matrix}\right.\right] dy$$

$$=\frac{2^{a}\,\{4(1+t)\}^{-\rho}\;\Gamma(1+a/2)\,\Gamma(1+a-b)\,\Gamma(\mu)}{\sqrt{\pi}\;\;\Gamma(1+a)\,\Gamma(1+a/2-b)}\sum_{u=0}^{[n/m]}\frac{(-n)_{mn}\;\;c^{u}}{u!\;\;\{4(1-t)\}^{u\gamma}}$$

$$\cdot A_{n,\,u}\,H_{P+2,\,Q+2}^{M+2,\,N+1}\left[z\{4(1+t)^{-T}\left|\begin{matrix}A'''\\ B'''\end{matrix}\right.\right], \qquad \ldots(3.4)$$

where

$$A'''=(1-\rho-\gamma\, u,\,T),\,(a_j,\,a_j)_{1,\,P},\,(1+a-b-\rho-\gamma\, u,\,T)$$

$$B'''=(\tfrac{1}{2}+a/2-\rho-\gamma u,\,T),\,(1+a/2-b-\rho-\gamma\, u,\,T),\,(b_j,\,\beta_j)_{1,\,Q}.$$

The conditions of validity of the integral (3.1) through (3.4) can be deduced easily from the conditions mentioned with the main result (2.1).

The above integral (3.4) is a special case of the integral evaluated recently by Saxena and Ram [4, *p.* 23 Eq. (3.1)]. Further, taking $n=0$, we obtain another known result due to Rathie [3, *p.* 255, Eq. (2.4)].

Lastly, we observe that on following the same lines as indicated avove, a number of integrals can be obtained easily from our integrals (2.2) and (2.3) as special cases.

ACKNOWLEDGEMENTS

The author is grateful to Dr. S. P. Goyal, University of Rajasthan, Jaipur, for help and able guidance during the present work. He is also thankful to Professor H. M. Srivastava, University of Victoria, Canada, for his certain useful suggestions and encouragement.

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Jñānabha, Vol, 20, 1990

On Integrals Involving Genralised Polynomials

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(Recived : May 2, 1989, Revised : Novemeber 29, 1989)

ABSTRACT

Eight integral formulas involving generalised Gould-Hopper polynomials and Jacobi polynomials are established here. The technique employed is that of simple differentiation of certain integrals established in Section 2.

1. INTRODUCTION

Recently, Kalla [2] evaluated certain integrals involving Jacobi polynomials and Jacobi functions and their partial derivatives. In this paper we establish eight integral formulas involving the generalised Gould-Hopper polynomials and Jacobi polynomials by differentiating partially the two integrals established first in Section 2 below. A more general polynomial than the Gould-Hopper polynomials was studied earlier by Brafman [4, *p.* 76, Eq. (8)] in the following form :

$$\underset{n}{B}^{m}[(a_r);(\beta_s):x]$$

$$={}_{m+r}F_s\ [\triangle(m;-n),a_1,\ldots,a_r;\beta_1,\ldots,\beta_s:x]$$

$$=n!\sum_{k=0}^{[n/m]}\left[\frac{(-1/m)^{mk}}{(n-mk)!}\ \frac{\prod_{j=1}^{r}(a_j)_k}{\prod_{j=1}^{s}(\beta_j)_k}\ \frac{x^k}{k!}\right] \qquad \ldots(1.1)$$

where $\triangle\,(m;\lambda)$ abbreviates the array of m parameters :

$\frac{\lambda}{m},\ \frac{\lambda+1}{m},\ldots,\ \frac{\lambda+m-1}{m}$ $m\geqslant 1$ and $[n/m]$ is the greatest integer in n/m. When $r=0$ and $s=0$, the class of polynomials in (1.1) reduces essentially to the Gould-Hopper polynomials.

2. INTEGRALS REQUIRED

The following two integrals are required to evaluate the main integrals in the next section :

(*i*) For Re $(a) > -1$, Re $(b) > -1$

$$\int_{-1}^{1} (1-x)^{a}\,(1+b)^{b}\, B_{n}^{m}\left[(\alpha_r);(\beta_s) : \delta\left(\frac{1-x}{2}\right)^{\mu}\left(\frac{1+x}{2}\right)^{\nu}\right] dx$$

$$= n\,!\,2^{a+b+1} \sum_{k=0}^{[n/m]} \frac{(-1/m)^{mk}\,\delta^{k}}{k\,!\,(n-mk)\,!} A_k, \qquad \ldots(2.1)$$

where

$$A_k = \frac{\prod_{j=1}^{r} (\alpha_j)_k}{\prod_{j=1}^{s} (\beta_j)_k}\; B\,(a+\mu k+1,\, b+\nu k+1)\,; \qquad \ldots(2.2)$$

(*ii*)

$$I_{\alpha,\beta\,k}^{a,\,b} = \int_{-1}^{1} (1-x)^{a}\,(1+x)^{b}\, P_{k}^{(\alpha,\,\beta)}(t+\rho)\; P_{k}^{(\alpha,\,\beta)}(t-\rho)\, dx$$

$$= c_k\, 2^{a+b+1}\, B\,(a+1,\, b+1) \sum_{r=0}^{k} \frac{(-k)_r\,(\alpha+\beta+k+1)_r}{(\beta+1)_r}$$

$$\cdot\; {}_3F_2\left[\begin{matrix} -r,\ \alpha+\beta+r+1,\ a+1 \\ a+1,\ a+b+2 \end{matrix}\ ;\ 1\right]\frac{t^{r}}{r\,!} \qquad \ldots(2.3)$$

$$= c_k\, 2^{a+b+1} \sum_{r=0}^{k} \sum_{s=0}^{r} \frac{(-k)_r\,(-r)_s}{s\,!}\, b_{r,\,s}\, \frac{t^{r}}{r\,!}, \qquad \ldots(2.4)$$

where

$$c_k = \frac{(-1)^{k}\,(\alpha+1)_k\,(\beta+1)_k}{(k\,!)^{2}} \qquad \ldots(2.5)$$

$$b_{r,s} = \frac{(\alpha+\beta+k+1)_r\,(\alpha+\beta+r+1)_s}{(\beta+1)_r\,(\alpha+1)_s}\; B\,(b+1,\, a+s+1), \qquad \ldots(2.6)$$

$P_{n}^{(\alpha,\,\beta)}(x)$ is the familiar Jacobi polynomial [3, *p*. 254, Eq. (1), (3)],

and $\rho = (1-2xt+t^{2})^{1/2}$. ...(2.7)

Formula (2.1) is easily established by writing the series form of the polynomials $B_{n}^{m}\;[(\alpha_r);(\beta_s) : \delta\left(\frac{1-x}{2}\right)^{\mu}\left(\frac{1+x}{2}\right)^{\nu}]$ from (1.1) and then integrating term-by-term with the help of the result [3, *p*. 31, Eq. (7)], viz

$$\int_{-1}^{1} (1-x)^{a-1}\,(1+x)^{b-1}\, dx = 2^{a+b-1}\, B\,(a,\, b), \quad \text{Re}\,(a) > -1, \quad \text{Re}(b) > -1 \qquad \ldots(2.8)$$

Formula (2.3) is similarly established by first using the identity

$$P_k^{(\alpha,\beta)}(t+\rho)\,P_k^{(\alpha,\beta)}(t-\rho)$$

$$=\frac{(-1)^k(\alpha+1)_k(\beta+1)_k}{(k!)^2}\sum_{r=0}^{k}\frac{(-k)_r(\alpha+\beta+1)_r\,P_r^{(\alpha,\beta)}(x)}{(\alpha+1)_r(\beta+1)_r}t^r \quad \ldots(2.9)$$

in the integrand, integrating term-by-term, and then using the result [1, *p.* 841, Eq. 7.391 (2)], viz

$$\int_{-1}^{1}(1-x)^a(1+x)^b\,P_r^{(\alpha,\beta)}(x)\,dx$$

$$=2^{a+b+1}B(a+1,\,b+1)\frac{(\alpha+1)_r}{r!}\;{}_3F_2\left[\begin{matrix}-r,\,\alpha+\beta+r+1,\,a+1\\ \alpha+1,\,a+b+2\end{matrix};1\right]. \quad \ldots(2.10)$$

Formula (2.3) is obtained from (2.3) by using the series form of ${}_3F_2$.

3. MAIN INTEGRALS

Let

$$F(x)=(1-x)^a(1+x)^b\,B_n^m\left[(\alpha_r),(\beta_s);\;\delta\left(\frac{1-x}{2}\right)^\nu\left(\frac{1+x}{2}\right)^\mu\right],$$

$$G(x)=(1-x)^a(1+x)^b\,P_k^{(\alpha,\beta)}(t+\rho)\,P_k^{(\alpha,\beta)}(t-\rho),$$

and $\rho=(1-2xt+t^2)^{1/2}$.

If $\psi(z)$ denotes the logarithmic derivative of the gamma function $\Gamma(z)$ *i.e.*, $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$, then, with A_k, c_k and $b_{r,s}$ given by equations (2.2), (2.5)

$$\int_{-1}^{1}F(x)\log(1-x)\,dx$$

$$=n!\,2^{a+b+1}\sum_{k=0}^{[n/m]}\frac{(-1/m)^{mk}\,\delta^k}{k!\,(n-mk)!}A_k\,[\log 2+\psi(a+\nu k+1)$$

$$-\psi(a+b+(\mu+\nu)k+2)] \quad \ldots(3.2)$$

$$\int_{-1}^{1}F(x)\log(1-x^2)\,dx$$

$$=n!\,2^{a+b+1}\sum_{k=0}^{[n/m]}\frac{(-1/m)^{mk}\,\delta^k}{k!\,(n-mk)}A_k\,[\log 4+\psi(a+\mu k+1)$$

$$+\psi(b+\nu k+1)-2\psi(a+b+(\mu+\nu)k+2)] \quad \ldots(3.3)$$

$$\int_{-1}^{1}F(x)\log\left(\frac{1-x}{1+x}\right)dx$$

$$= n!\, 2^{a+b+1} \sum_{k=0}^{[n/m]} \frac{(-1/m)^{mk}\,\delta^k}{k!\,(n-mk)!}\, A_k\, [\psi\,(a+\mu k+1)-\psi\,(b+\nu k+1)] \qquad \ldots(3.4)$$

$$\int_{-1}^{1} G(x)\log(1-x)\,dx$$

$$= (\log 2)\; I^{a,\,b}_{\alpha,\,\beta,\,k} + c_k\, 2^{a+b+1} \sum_{r=0}^{k} \sum_{s=0}^{r} \frac{(-k)_r}{r!}\,\frac{(-r)_s}{s!}$$

$$\cdot\, b_{r,\,s}\, t^r\, [\psi\,(a+s-1)-\psi\,(a+b+s+2)] \qquad \ldots(3.5)$$

$$\int_{-1}^{1} G(x)\log(1+x)\,dx$$

$$= (\log 2)\; I^{a,\,b}_{\alpha,\,\beta,\,k} + c_k\, 2^{a+b+1} \sum_{r=0}^{k} \sum_{s=0}^{r} \frac{(-k)_r}{r!}\,\frac{(-r)_s}{s!}$$

$$.\; b_{r,\,s}\, t^r\, [\psi\,(b+1)-\psi\,(a+b+s+2)] \qquad \ldots(3.6)$$

$$\int_{-1}^{1} G(x)\log(1-x^2)\,dx$$

$$= (\log 4)\; I^{a,\,b}_{\alpha,\,\beta,\,k} + c_k\, 2^{a+b+1} \sum_{r=0}^{k} \sum_{s=0}^{r} \frac{(-k)_r}{r!}\,\frac{(-r)_s}{s!}$$

$$\cdot\, b_{r,\,s}\, t^r\, [\psi\,(a+s+1)+\psi\,(b+1)-2\psi\,(a+b+s+2\ ] \qquad \ldots(3.7)$$

$$\int_{-1}^{1} G(x)\log\left(\frac{1-x}{1+x}\right)dx$$

$$= c_k\, 2^{a+b+1} \sum_{r=0}^{k} \sum_{s=0}^{r} \frac{(-k)_r\,(-r)_s}{r!\quad s!}\, b_{r,\,s}\, t^r\, [\psi\,(a+s+1)-\psi\,(b+1)] \qquad \ldots 3.8)$$

Outline of proof : The result in (3.1) and (3.2) are established easily by taking the partial derivatives of (2.1) with respect to a and b respectively. We next get the integrals in (3.3) and (3.4) by first adding (3.1) and (3.2). and then subtracting them. The results in (3.5) to (3.8) are established similarly from (2.4).

4. PARTICULAR CASES

First we take $r=0$, $s=1$ and $\beta_1=1+a$, $m=1$ in (1.1), it yields the Laguerre polynomial $L_n^{(a)}(x)$ [3, *p.* 200, Eq. (1); we have the relation

$$B_n^1\,[-;\,1+a;\,x]$$

$$={}_1F_1\,(-n;\,1+\alpha;\,x]=\frac{n\,!}{(1+\alpha)_n}\;L_n^{(\alpha)}\,(x), \qquad \text{...(4.1)}$$

On making these changes in (3.3), we get the following integral involving the Laguerre polynomials :

$$\int_{-1}^{1}(1-x)^a\,(1+x)^b\log\,(1-x^2)\,L_n^{(\alpha)}\left[\,\delta\left(\frac{1-x}{2}\right)^{\mu}\left(\frac{1+x}{2}\right)^{\nu}\right]dx$$

$$=(1+\alpha)_n\,2^{a+b+1}\sum_{k=0}^{n}\frac{(-\delta)^k}{k\,!\,(n-k)!}\;\frac{B\,(a+\mu k+1,\,c+\nu k+1)}{(1+\alpha)_k}$$

$$\cdot\,[\log 4+\psi\,(a+\mu k+1)+\psi\,(b+\nu k+1)-2\psi\,(a+b+(\mu+\nu)\,k+2)]. \qquad \text{...(4.2)}$$

Next, if we take $\alpha=\beta=0$, $P_n^{(\alpha,\,\beta)}\,(x)$ reduces to a Legendre polynomial and we accordingly have, from (3.8), the following result involving Legendre polynomials :

$$\int_{-1}^{1}(1-x)^a\,(1+x)^b\,\log\left(\frac{1-x}{1+x}\right)P_k\,(t+\rho)\,P_k\,(t-\rho)\,dx$$

$$=(-1)^k\,2^{a+b+1}\sum_{r=0}^{k}\sum_{s=0}^{r}\frac{(-k)_r\,(-r)_s\,(1+k)_r\,(1+r)_s}{(r\,!)^2\qquad(s\,!)^2}$$

$$\cdot\;B\,(a+s+1,\,b+1)\,[\psi\,(a+s+1)-\psi\,(b+1)]. \qquad \text{...(4.3)}$$

Results similar to these ones can also be deduced from equations **(3.1)** (3.2), (3.4), and (3.5), (3.6), (3.7), but are not recorded here for lack of space.

Finally, if we take $b=\beta$ in (2 3), it gives :

$$I_{\alpha,\,\beta,\,k}^{a,\,b}=\int_{-1}^{1}(1+x)^a\,(1+x)^b\;P_k^{(\alpha,\,\beta)}\,(t+\rho)\;P_k^{(\alpha,\,\beta)}\;(t-\rho)\,dx$$

$$=c_k\;2^{a+b+1}\sum_{r=0}^{k}\frac{(-k)_r\,(\alpha+\beta+k+1)_r\,(a-\alpha)_r}{(1+\alpha)_r\qquad(1+\beta)_r}B(a+1,\,\beta+r+1)\,\frac{t^r}{r!} \qquad \text{...(4.4)}$$

Differentiating (4.4) partially with respect to a and then comparing the result thus obtained with that in (3.5) (with $b=\beta$ therein), if we equate the coefficients of t^r on both sides, we get a known result due to Kalla [2, p, 128, Eq. (23)].

ACKNOWLEDGEMENTS

The authors are grateful to Professor H.M. Srivastava for his kind suggestions that resulted in the present from of the paper.

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Jñānābha, Vol. 20, 1990

Associated Operational Calculus For a Bessel-Clifford Operator

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(Received : December 12, 1988, Revised : January 12, 1990)

ABSTRACT

In this paper an operational calculus for a certain Bessel-Clifford operator is developed by following an algebraic process similar to Mikusinski's. To achieve this a new convolution is introduced, which proves an extension to Koh's. Two applications derived from the ensuing calculus are shown.

1. INTRODUCTION

Ditkin [2], followed further by Ditkin and Prudnikov [3], has developed and operational calculus for the DtD-operator, through an algebraic process similar to Mikusinski's.

This calculus has been extended by Meller [10] to the operator $B_\nu = t^{-\nu} D t^{\nu+1} D$, when $-1 < \nu < 1$, and recently Koh [6], on using the Riemann-Liouville fractional derivative, extends this operational calculus (for the B_ν-operator) to the interval $-1 < \nu < +\infty$. The results obtained in [6] include those given in [3] for $\nu = 0$ and in [10] for $\nu \in (-1, 1)$.

In this paper an operational calculus for the operator $A_{\alpha,\beta} = t^{1-\alpha-\beta} D t^{\alpha} D t^{\beta}$, $\alpha \geqslant 1$ and $\beta \geqslant 0$, has been developed, which generalizes the previous work of Koh when $\alpha = \nu + 1$ and $\beta = 0$ [6], and the results of Ditkin [2] and Ditkin-Prudnikov [3], for $\alpha = 1$ and $\beta = 0$, as well.

2. THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRATION AND DIFFERENTIATION

The Riemann-Liouville integral or order $\nu > 0$, is defined as (see [14]) :

$$I^{\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} f(t)\, dt \qquad \ldots(2.1)$$

Some well-known properties of this operator are as follows :

$$I^0 f(t) = \lim_{\nu \to 0} I^\nu f(t) = f(t) \qquad \ldots(2.2)$$

$$D I^{\nu+1} f(t) = I^\nu f(t), \ \nu > 0 \qquad \ldots(2.3)$$

$$I^\nu I^\mu f(t) = I^{\nu+\mu} f(t), \ \nu, \mu > 0 \qquad \ldots(2.4)$$

$$I^\nu t^k = \frac{\Gamma(k+1)}{\Gamma(\nu+k+1)} t^{\nu+k} \ (k+1 > 0, \ \nu \geqslant 0) \qquad \ldots(2.5)$$

whereas the Riemann-Liouville fractional derivative of order $\nu > 0$ of a function $f(t) \epsilon C^n$ $((0, \infty))$ is defined to be $D^\nu f(t) = D^n I^{n-\nu} f(t)$ $(n-1 < \nu \leqslant n)$.

From (2.1) the following properties can be derived :

$$D^\nu I^\mu f(t) = I^{\mu-\nu} f(t) \quad (\nu, \mu, \mu-\nu \epsilon K^* \cup \{0\}) \qquad \ldots(2.6)$$

$$D^\nu t^k = \frac{\Gamma(k+1)}{\Gamma(k-\nu+1)} t^{k-\nu} \quad (-1 < k < \infty, \ 0 < \nu \leqslant k) \qquad \ldots(2.7)$$

$$D^\nu f(t) = I^{-\nu} f(t). \qquad \ldots(2.8)$$

3. THE EXTENSION OF C^∞ TO THE QUOTIENT FIELD

Let α and β be real numbers with $\alpha \geqslant 1$ and $\beta \geqslant 0$. Consider the set of functions denoted as C^∞ $([0, \infty))$, where the following operation is defined :

$$f(t)^* g(t) = \frac{1}{\Gamma(\alpha+\beta)\Gamma(\beta+1)} t^{-\beta} D^{\beta+1} t^{2-\alpha+\beta} D^{\alpha+\beta} \cdot$$
$$\cdot \int_0^t (t-\xi)^{\alpha+\beta-1} \xi^{\alpha+\beta-1} d\xi \int_0^1 \eta^\beta (1-\eta)^\beta f(\xi\eta) g[(1-\eta)(t-\xi)] d\eta.$$

for any pair $f(t), g(t) \epsilon C^\infty$ . ...(3.1)

From the very definitions for the fractional derivative and integral it can be easily seen that :

$$t^p * t^q = \frac{\Gamma(\beta+p+1)\,\Gamma(\beta+q+1)\,\Gamma(\alpha+\beta+p)\,\Gamma(\alpha+\beta+q)}{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)\,\Gamma(\alpha+\beta+p+q)\,\Gamma(\beta+p+q+1)} t^{p+q} \quad (3.2)$$

holds for $p, q \epsilon N \cup \{0\}$.

Hence, by invoking Weierstrass's approximation theorem (3.2) can be proved for any pair $f(t)$, $g(t)$ of functions in C^∞ . Then, the *-operation is closed on C^∞ . Moreover, if $f(t)$, $g(t)$, $h(t)$ belong to C^∞ , then the following list of properties holds :

(*i*) $f(t)^* g(t) = g(t)^* f(t)$
(*ii*) $f(t)^*(g(t)^* h(t)) = (f(t)^* g(t))^* h(t)$
(*iii*) $f(t)^*(g(t) + h(t)) = f(t)^* g(t) + f(t)^* h(t)$
(*iv*) $A_{\alpha,\beta}(f(t)^* g(t)) = (A_{\alpha,\beta} f(t))^* g(t) = f(t)^*(A_{\alpha,\beta}\, g(t))$

Note 1. Since (*iv*) holds, we shall say according to Dimovski [1], that the sign* stands for a convolution for the operator $A_{\alpha,\beta}$ on C^∞ .

Proposition 1. For any complex number λ and any $f(t)\epsilon C^{\infty}$, we have

$$\lambda^* f(t)=\lambda f(t).$$

Proof. From the previous, the following can be inferred.

$$\lambda^* f(t)=\frac{\lambda}{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}\, t^{-\beta} D^{\beta+1} t^{2-\alpha+\beta} D^{\alpha+\beta}.$$

$$\int_0^t (t-\xi)^{\alpha+\beta-1} \xi^{\alpha+\beta-1} d\xi \int_0^1 \eta^{\beta}\,(1-\eta)^{\beta} f(\xi\eta) d\eta$$

$$=\lambda t^{-\beta} D^{\beta+1} t^{2-\alpha+\beta} D^{\alpha+\beta} I^{\alpha+\beta} t^{\alpha-\beta-2} I^{\beta+1} t^{\beta}\; f(t)=\lambda\, f(t).$$

Proposition 2. If $f(t)^* g(t)=0,\ f(t),\ g(t)\epsilon C^{\infty}$, then either of these functions equals zero, i. e. C^{∞} has no zero divisors.

Proof. Set $f(t)^* g(t)=0$. Since $t^{-\beta}\neq 0$ $(0, \infty)$, then the following can be established

$$D^{n+1} I^{n-\beta} t^{2-\alpha+\beta} D^{\alpha+\beta} \int_0^t (t-\xi)^{\alpha+\beta-1} \xi^{\alpha+\beta-1}.$$

$$\int_0^1 \eta^{\beta}(1-\eta)^{\beta} f(\xi\eta) g[(1-\eta)\;(t-\xi)]\, d\eta\, d\xi=0.$$

Hence

$$\int_0^t (t-\xi)^{n-\beta-1} \xi^{2-\alpha+\beta} D^{\alpha+\beta} \int_0^{\xi} (\xi-\eta)^{\alpha+\beta-1} \eta^{\alpha+\beta-1}.$$

$$\cdot\int_0^1 x^{\beta}\,(1-x)^{\beta} f(\eta x) g[(1-\lambda)(\xi-\eta)]\, dx\, d\eta\, d\xi=$$

$$=C_1\frac{t^n}{n!}+C_2\frac{t^{n-1}}{(n-1)!}+\dots+C_n t+C_{n+1} \qquad \dots(3.3)$$

is true.

As a consequence, $C_{n+1}=0$ for $t=0$. But it is well-known that if $C_1\neq 0$ for some $i\epsilon(1, 2, \dots n)$, then $f(t)$ and $g(t)$ are polynomials and in this case it follows from Weierstrass's approximation theorem that the left side of (3.3) is of a degree greater than n. Thus, the right side of (3.3) must be zero.

Now, by following a similar line of reasoning and by invoking further Titchmarsh's theorem, we can state that

$$\int_0^t \int_0^1 (t-\xi)^{\alpha+\beta-1}\; \xi^{\alpha+\beta-1}\; x^{\beta}\,(1-x)^{\beta}\; f(\xi x) g[(1-x)\;(t-\xi)]\, dx\; d\xi=0 \quad (3.4)$$

holds.

Now, if we perform in (3.4) the following change of variables $x=\frac{y}{\tau}$, $\xi=z\tau, t=v\tau$, we obtain

$$\int_0^v \int_0^\tau \tau^{a-2} z^{a+\beta-1} y^\beta f(zy) (\tau-y)^\beta (v-z)^{a+\beta-1} g[(\tau-y)(v-z)]\, dy\, dz=0. \qquad \ldots(3.5)$$

and then from Mikusinski's theorem and Ryll's-Nardzewski's as well, [12], it can be finally inferred that either of the following equations holds :

$$z^{a+\beta-1} y^\beta f(zy)=0 \text{ or } (\tau-y)^\beta (v-z)^{a+\beta-1} g[(\tau-y)(v-z)=0$$

Hence, either $f(t)$ is $g(t)$ is zero, which completes the proof.

By virtue of proposition 1, it follows that the unit element for * in C^∞ is the plain number 1.

We can therefore establish the following :

Proposition 3. The set C^∞ endowed with the ordinary sum and * as multiplication, becomes a unitary commutative ring without zero divisors, which can be extended to a quotient field

$$M=C^\infty \times (C^\infty - \{0\})/\sim,$$

where equivalence $\sim$ is defined on $C^\infty \times (C^\infty - \{0\})$ in the usual form, say

$$(f(t), g(t)) \sim (\bar{f}(t), \bar{g}(t)) \Leftrightarrow f(t) * \bar{g}(t) = g(t) * \bar{f}(t).$$

From now on, the pair $(f(t), g(t))$ will be noted as $\frac{f(t)}{g(t)}$ and will be regarded as an operator.

Define on M the usual operations of addition, multiplication and multiplication by scalar, as :

$$\frac{f(t)}{g(t)} + \frac{\bar{f}(t)}{\bar{g}(t)} = \frac{f(t) * \bar{g}(t) + \bar{f}(t) * g(t)}{g(t) * \bar{g}(t)}$$

$$\frac{f(t)}{g(t)} \cdot \frac{\bar{f}(t)}{g(t)} = \frac{f(t) * \bar{f}(t)}{g(t) * \bar{g}(t)}$$

$$\lambda \cdot \frac{f(t)}{g(t)} = \frac{\lambda f(t)}{g(t)}$$

Thus, M becomes an algebra.

The quotient set M contains a subset M', which is isomorphic to C^∞, through the following application

$$M' \subset M \to C^\infty$$

$$\frac{f(t)}{1} \to f(t).$$

Hence, operators of the form $\frac{f(t)}{1}$ constitute a subring of M.

4. AN OPERATIONAL CALCULUS

Let us now prove that the operator $A_{\alpha,\beta}$ belongs to M. First of all, let us state the following,

Proposition 4. The operator $A_{\alpha,\beta}$ can be expressed as :

$$A_{\alpha,\beta}=t^{1-\alpha-\beta}Dt^{\alpha}Dt^{\beta}=t^{-\beta}Dt^{2-\alpha}Dt^{\alpha+\beta-1} \qquad \text{...(4.1)}$$

Proof. Both operators reduce to the differential operator,

$$tD^2+(\alpha+2\beta)D+t^{-1}\beta(\alpha+\beta-1).$$

On the other hand the right hand inverse operator for $A_{\alpha,\beta}$ is given as:

$$R_{\alpha,\beta}\,f(t)=t^{-\beta}\int_0^t \xi^{-\alpha}d\xi\int_0^{\xi}\eta^{\alpha+\beta-1}f(\eta)d\eta=t^{1-\alpha-\beta}\int_0^t\xi^{\alpha-2}d\xi\int_0^{\xi}\eta^{\beta}f(\eta)d\eta.$$

due to the fact that $A_{\alpha,\beta}\cdot R_{\alpha,\beta}\,f(t)=f(t)$ is true for each $f(t)\epsilon C^{\infty}$.

Now, if we restrain the domain of the operator $A_{\alpha,\beta}$ to be the set.

$$\{f(t)/f(t)\epsilon C^{\infty},\ t^{\beta}f(t).\mid_{t=0^+}=0\},$$

then $R_{\alpha,\beta}$ becomes a left-hand inverse operator for $A_{\alpha,\beta}$. That is to say,

$$R_{\alpha,\beta}\,A_{\alpha,\beta}\,f(t)=f(t).$$

Proposition 5. Equality

$$\frac{t}{(\alpha+\beta)\,(\beta+1)}*f(t)=R_{\alpha,\beta}\,f(t) \text{ holds for any } f(t)\epsilon C^{\infty}.$$

Proof. Starting from

$$\frac{t}{(\alpha+\beta)\,(\beta+1)}*f(t)=\frac{1}{\Gamma(\alpha+\beta+1)\,\Gamma(\beta+2)}t^{-\beta}D^{\beta+1}t^{2-\alpha+\beta}D^{\alpha+\beta}$$
$$\int_0^t(t-\xi)^{\alpha+\beta}\xi^{\alpha+\beta-1}\,d\xi\cdot\int_0^1\eta^{\beta}(1-\eta)^{\beta+1}f(\xi\eta)\,d\eta.$$

and by performing the change of variable $\xi\eta=x$, we obtain :

$$t^{-\beta}D^{\beta+1}t^{2-\alpha+\beta}D^{\alpha+\beta}I^{\alpha+\beta+1}t^{\alpha\cdot\beta-3}I^{\beta+}t^{\beta}f(t)=$$
$$=t^{-\beta}II^{-\beta-2}t^{2-\alpha+\beta}I^{\alpha}I^{1-\alpha}t^{\alpha-\beta-3}I^{\beta+2}t^{\beta}f(t). \qquad \text{...(4.2)}$$

But since $-\beta-2+2-\alpha+\beta+\alpha=0$, it can be stated that (see [9])

$$I^{-\beta-2}t^{2-\alpha+\beta}I^{\alpha}=t^{-\alpha}I^{\alpha-\beta-2}t^{\beta+2}.$$

As a consequence, (4.2) can be written as

$$t^{-\beta}It^{-\alpha}I^{\alpha-\beta-2}t^{\beta+2}I^{1-\alpha-\beta-3}I^{\beta+2}t^{\beta}f(t)=$$
$$=t^{-\beta}It^{-\alpha}II^{\alpha-\beta-3}t^{\beta+2}I^{1-\alpha}t^{\alpha\ \beta-3}I^{\beta+2}t^{\beta}f(t). \qquad \text{...(4.3)}$$

And now by applying once more the second index law [9] to

$$I^{\alpha-\beta-8}t^{\beta+2}I^{1-\alpha}=t^{\alpha-\beta-2}t^{3+\beta-\alpha},$$

then (4.) becomes

$$t^{-\beta}\, It^{-\alpha}\, It^{\alpha-1} I^{-\beta-2} t^{3+\beta-2}\, t^{\alpha-\beta-3}\, I^{\beta+2} t^{\beta}\, f(t)$$

$$=t^{-\beta} It^{-\alpha}\, It^{\alpha+\beta-1} f(t)=t^{-\beta}\int_0^t \xi^{-\alpha}\, d\xi \int_0^{\xi} \eta^{\alpha+\beta-1} f(\eta)\, d\eta.$$

From this result the following can be established by induction.

Prodosition 6. Letting $f(t)\epsilon C^{\infty}$ and $k\epsilon N$, then the following holds.

$$\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+k)\,\Gamma(\beta+k+1)}\, t^{k*} f(t)=R^{k}_{\alpha,\,\beta} f(t).$$

Proof. The previous expression has already been verified for the case $k=1$.

Suppose now that it is valid for $k=m$ and let us see that it also holds for $k=m+1$

$$\frac{t}{(\alpha+\beta)\,(\beta+1)} * \left(\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+m)\,\Gamma(\beta+m+1)}\, t^{m*} f(t)\right)$$

$$=R_{\alpha,\beta}\left(R^{m}_{\alpha,\,\beta}\, f(t)\right)$$

$$\Rightarrow\left(\frac{t}{(\alpha+\beta)\,(\beta+1)} * \frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+m)\,\Gamma(\beta+m+1)}\, t^{m}\right) * f(t)$$

$$=R^{m+1}_{\alpha,\,\beta}\, f(t)$$

which reduces to

$$\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+m+1)\,\Gamma(\beta+m+2)}\, t^{m+1} * f(t)=R^{m+1}_{\alpha,\,\beta} f(t).$$

Hence, the operators $R^{k}_{\alpha,\,\beta}$ belong to M and can be therefore identified with the following expression :

$$\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+k)\,\Gamma(\beta+k+1)}\, t^{k}.$$

Besides, the following can be established :

Proposition 7. Equality

$$f(t)=R_{\alpha,\,\beta}\, A_{\alpha,\,\beta} f(t)+t^{-\beta}[t^{\beta} f(t)]\Big|_{t=0^{+}}. \qquad \ldots(4.4)$$

holds for any $f(t)\epsilon C^{2}([0,\,\infty))$

Proof. Suffice it to consider that

$$R_{\alpha,\beta}\; R_{\alpha,\beta}\; f(t)=t^{-\beta}\int_0^t \xi^{-\alpha}\, d\xi\int_0^{\xi} \eta^{\alpha+\beta-1}\,(A_{\alpha,\,\beta} f(\eta)\, d\eta$$

$$=t^{-\beta}\int_0^t \xi^{-\alpha} d\xi\int_0^{\xi} D\eta^{\alpha}\; D\eta^{\beta} f(\eta)\, d\eta=t^{-\beta}\int_0^t D\xi^{\beta} f(\xi)$$

$$=f(t)-t^{-\beta}\,[t^{\beta} f(t)]\Big|_{t=0^{+}}$$ is true.

Now if we set

$$V=\frac{(\alpha+\beta)\,(\beta+1)}{t}, \qquad \ldots(4.5)$$

then the following can be verified

Proposition 8.

$$Vf(\mathrm{t})=A_{\alpha,\,\beta}\,f(t)+t^{-\beta}\,[t^{\beta}f(t)]_{|t=0^+}V. \qquad \ldots(4.6)$$

holds for any $f(t)\in C^2\,([0,\,\infty))$

Proof. It is known that

$$f(t)=R_{\alpha,\,\beta}\,A_{\alpha,\,\beta}f(t)+t^{-\beta}\,[t^{\beta}f(t)]_{|t=0^+}$$

$$=\frac{t}{(\alpha+\beta)\,(\beta+1)}*A_{\alpha,\,\beta}f(t)+t^{-\beta}\,[t^{\beta}f(t)]_{|t=0^+}$$

then

$$Vf(t)=\left(\frac{(\alpha+\beta)\,(\beta+1)}{t}*\frac{t}{(\alpha+\beta)(\beta+1)}\right)*A_{\alpha,\,\beta}\,f(t)+t^{-\beta}\,[t^{\beta}f(t)]_{|t=0^+}V$$

$$=A_{\alpha,\,\beta}\,f(t)=t^{-\beta}\,[t^{\beta}f(t)]_{|t=0^+}V.$$

5. OPERATIONAL RULES

Consider the two functions

$$(i)\quad C_{\alpha-1,\,\beta}(t)=\sum_{r=0}^{\infty}\frac{(-1)^r\,\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\beta+r+1)\,\Gamma(\alpha+\beta+r)}t^r \qquad \ldots(5.1)$$

$$(ii)\quad E_{\alpha-1,\,\beta}(t)=\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+\beta)\,\Gamma(\beta+1)}{\Gamma(\beta+r+1)\,\Gamma(\alpha+\beta+r)}t^r \qquad \ldots(5.2)$$

we can formally establish that in $(0,\,\infty)$

$$t^{-\beta}*C_{\alpha-1,\,\beta}(at)=\Gamma(\alpha)t^{-\beta}\,C_{\alpha-1}\,(at) \qquad \ldots(5.3)$$

$$t^{-\beta}*E_{\alpha-1,\,\beta}(at)=\Gamma(\alpha)t^{-\beta}\,E_{\alpha-1}(at) \qquad \ldots(5.4)$$

where $C_{\alpha-1}\,(at)$ and $E_{\alpha-1}\,(at)$ are respectively the Bessel-Clifford function of first kind and the modified of first kind, both of them of order $\alpha-1$[5].

In the following some operational rules are obtained, which will come in useful to certain applications.

As it can be easily proved, the differential equations

$$(A_{\alpha,\,\beta} \pm a)y=0 \quad (a>0) \qquad \dots(5.5)$$

have the following functions as solution :

$$y_1(t)=2^{1-\alpha}\, t^{-\beta}\, C_{\alpha_{-1}}(at) \qquad \text{(for the plus-sign)}$$

$$y_2(t)=2^{1-\alpha}\, t^{-\beta}\, E_{\alpha_{-1}}(at) \qquad \text{(for the minus-sign).}$$

Since

$$\lim_{t\to 0+} t^{\beta}\, y_1(t)=\lim_{t\to 0+} t^{\beta}\, y_2(t)=\frac{2^{1-\alpha}}{\Gamma(\alpha)} \quad \text{is true,}$$

then from 4.6) and (5.5) it can be inferred that

$$V^*f(t) = \mp a^*f(t)+t^{-\beta}\,[t^{\beta} f(t)]_{|t=0^+}V$$

$$\Rightarrow \begin{cases} (V+a)^*(t^{-\beta} * C_{\alpha-1,\,\beta}(at))=t^{-\beta} * V \\ (V-a)^*(t^{-\beta} * E_{\alpha-1,\,\beta}(at))=t^{-\beta} * V \end{cases}$$

which leads, among others, to the following operational rules :

(*a*) $$\frac{V}{V+a}=C_{\alpha-1,\,\beta}(at)$$

(*b*) $$\frac{V}{V-a}=E_{\alpha-1,\,\beta}(at)$$

(*c*) $$\frac{aV}{V^2-a^2}=\tfrac{1}{2}\left(E_{\alpha-1,\,\beta}(at)-C_{\alpha-1,\,\beta}(at)\right)$$

(*d*) $$\frac{V^2}{V^2-a^2}=\tfrac{1}{2}\left(E_{\alpha-1,\,\beta}(at)+C_{\alpha-1,\,\beta}(at)\right)$$

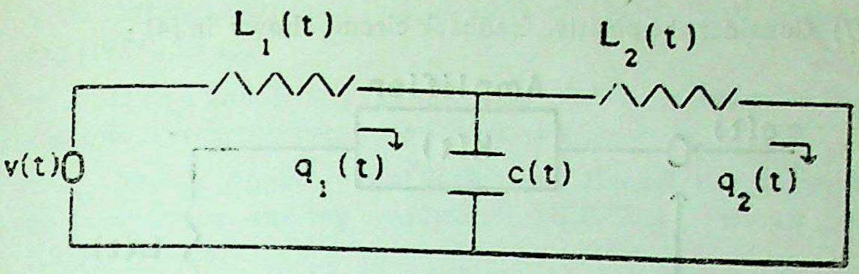
(*e*) $$\frac{a}{a-V}=1-E_{\alpha-1,\,\beta}(at)$$

(*f*) $$\frac{a}{a+V}=1-C_{\alpha-1,\,\beta}(a)$$

6. PRACTICAL APPLICATIONS

(I) Consider the electric network shown in [7], which is reproduced on next page.

This circuit is made up a time-variant source $v(t)$, two inductors *i.e.*, $L_1(t)$ and $L_2(t)$ and a capacitor $c(t)$. An adequate analysis yields the following system of equations :



(Fig. 1)

$$v(t)=DL_1(t)Dq_1(t)+\frac{q_1(t)-q_2(t)}{c(t)}$$

$$0=\frac{q_2(t)-q_1(t)}{c(t)}+DL_2(t)Dq_2(t).$$

If we assume that $L_1(t)=at^{\alpha}$ $L_2(t)=bt^{\alpha}$ and $c(t)=\frac{1}{c}t^{1-\alpha}$, with $a, b, c>0$ and set $q_i(t)=t^{\beta}\mu_i(t)$, $i=1, 2$, and $e(t)=t^{1-\alpha-\beta}v(t)$, it follows that

$$e(t)=aA_{\alpha,\beta}\ \mu_1(t)+c(\mu_1(t)-\mu_2(t))$$

$$0=c(\mu_2(t)-\mu_1(t))+bA_{\alpha,\beta}\mu_2(t).$$

Now, by invoking (4.6) and assuming $\mu_i(0)=0$, $i=1, 2$, the following can be obtained

$$e(t)=(aV+c)-c\mu_2(t)$$

$$0=c\mu_1(t)+(bV+c)\mu_2(t)$$

therefore,

$$\mu_1(t)=\frac{bV+c}{abV^2+c(a+b)V}\cdot e(t)$$

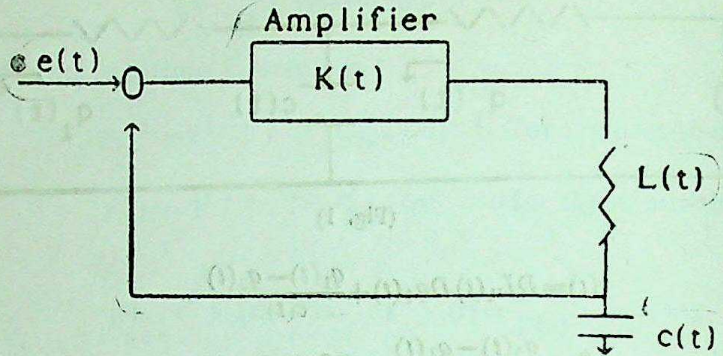
$$\mu_2(t)=\frac{c}{abV^2+c(a+b)V}.e(t).$$

Finally, the previous operational rules yield :

$$\mu_1(t)=\frac{1}{a+b}\left[\frac{b^2}{c(a+b)}-\frac{b^2}{c(a+b)}.C_{\alpha-1,\,\beta}\left(\frac{c(a+b)}{ab}t\right)\right.$$
$$\left.+\frac{t}{(\alpha+\beta)(\beta+1)}\right]*e(t)$$

$$\mu_2(t)=\left[\frac{ab}{c(a+b)}+\frac{ab}{c(a+b)}\cdot C_{\alpha-1,\,\beta}\left(\frac{c(a+b)}{ab}t\right)\right.$$
$$\left.+\frac{t}{(\alpha+\beta)(\beta+1)}\right]*e(t)$$

(*II*) Consider the positive feedback circuit shown in [4] :



(Fig. 2)

The electric charge $q(t)$ must satisfy the following equation :

$$DL(t)\,Dq(t)+\frac{q(t)}{c(t)}=k(t)\left(\ e(t)+\frac{q(t)}{c(t)}\right).$$

Suppose $k(t)=k$, $L(t)=at^{\alpha}$ and $c(t)=ct^{1-\alpha}$ with $a,\ b>0$. Then the previous expression reduces to

$$abA_{\alpha,\beta}\,q_1(t)+(1-k)q_1(t)=e_1(t),$$

where $q(t)=t^{\beta}q_1(t)$ and $e_1(t)=bke\,(t)t^{1-\alpha-\beta}$. As a consequence, if we demand that $q(0)=0$, then in can be easily obtained that

$$q_1(t)=\frac{1}{abV+(1-k)}.e_1(t)$$

and by invoking (4.5) and the rules (*e*) and (*f*) we obtain the following

$$q_1(t)=\frac{1}{k-1}\left[\,1-C_{\alpha-1,\ \beta}\left(\frac{1-k}{ab}t\ \right)\right]*e_1(t)\quad k\epsilon(0,\ 1)$$

$$q_1(t)=\frac{1}{ab}\ \frac{t}{(\alpha+\beta)\ (\beta+1)}*e_1(t)\qquad k=1$$

$$q_1(t)=\frac{1}{k-1}\left[\,1-E_{\alpha-1,\ \beta}\left(\frac{1-k}{ab}t\ \right]*e_1(t)\right.\quad k>1.$$

ACKNOWLEDGEMENTS

I am grateful to Professsor H. M. Srivastava of the University of Victoria for his helpful suggestions in the preparation of this paper. Finally, the author is also thankful to the referee for improvements in the paper.

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Jñānābha, Vol, 20, 1990

Petryshyn's Condition For Ky Fan Type Theorem*

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(Received : May 30, 1990)

0. INTRODUCTION

The following well-known theorem due to Ky Fan [3, Theorem 8] has applications in fixed point theory, approximation theory, minimax theory and avriational inequalities.

Let C be a nonempty compact convex subset of a normed linear space X and $f : C \to X$ a continuous function. Then there is a point $u \epsilon C$ such that

$$\| u - f(u) \| = d(f(u), C) := \inf \{ \| x - f(u) \| : x \epsilon C \}.$$

Lin [5, Theorem 2] proved a theorem of this type for a continuous set-condensing map defined on a nonempty closed bounded convex subset of a Hilbert space. A result analogous to that of Lin [5] was obtained by Singh and Watson [10] for nonexpansive maps.

Lin and Yen [6, Theorem 1] gave a common generalization of these results above replacing continuous set-condensing maps or nonexpansive maps, respectively, by continuous 1-set-contractive maps for which a certain range is closed. Obviously any set-condensing or non-expansive map is a 1-set-contractive map; further the closedness condition of [6, Theorem 1] is satisfied for a set-condensing or nonexpansive map f (defined on a closed convex bounded subset C of a Hilbert space) since in that case $(I - f)(C)$ is closed.

The condition that $(I - f)(C)$ is closed also appears in Browder's fixed point theorem [1, *p*. 230] to generalize the fixed point theorems given in ([4] or [9]) and [1, *p*. 104] for continuous set-condensing and nonexpansive maps, respectively.

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* This work was financed by funds of M. P. I. of Italy.

Petryshyn [8, Theorem 1] showed that in Browder's fixed point theorem for a map $f : C \to X$ the condition that $(I-f)(C)$ is closed can be replaced by the weaker condition

$$0 \in \overline{(I-f)(C)} \Rightarrow 0 \in (I-f)(C),$$

or equivalently, by
(*P*) if $\{x_n\}$ is any sequence in C such that $x_n - f(x_n) \to 0$ $(n \to \infty)$, then there exists $x \in C$ with $x = f(x)$.

In this note we show that several other theorems can be improved replacing the closedness condition by the condition (*P*) of Petryshyn, in particular [6, Theorem 1] of Lin and Yen mentioned above.

1. KY FAN TYPE THEOREM

We use the terminology of Petryshyn [8]. In the following let X be a Banach space and C be a nonempty closed convex subset of X.

Lemma 1. *Assume that $g : C \to C$ is a continuous 1-set-contractive map and $f(C)$ is bounded. Then there is a sequence $\{x_n\}$ in C such that $x_n - g(x_n) \to 0$ $(n \to \infty)$.*

Proof. Let $B := \overline{co}g(C)$ the closed convex hull of $g(C)$ and $x_n \in B$. Put $k_n = 1 - \frac{1}{n}$ and $g_n(x) = k_n g(x) + (1 - k_n)x_0$ for $x \in B$ and for all positive integers n. Then $g_n : B \to B$ is a continuous k_n-set-contractive map and therefore has by [4, Theorem 3] a fixed point $x_n \in B$. Since

$$x_n - g(x_n) = \frac{1}{n}\, g(x_n) - \frac{1}{n} x_0$$

and $\{g(x_n)\}$ is bounded, the sequence $\{x_n - g(x_n)\}$ converges to 0.

If in addition to the assumption of Lemma 1, the map g satisfies the following condition (*P*) of the introduction :

(*P*) If $\{x_n\}$ is any sequence in C such that $x_n - g(x_n) \to 0$ $(n \to \infty)$, then there exists $x \in C$ with $x = g(x)$,
then g has a fixed point by Lemma 1.

The following theorem generalizes [6, Theorem 1].

Theorem 1. *Let X be a Hilbert space, $f : C \to X$ be a continuous 1-set-contractive map such that $f(C)$ is bounded and assume that $p \circ f$ satisfies the condition (P), where p is the proximity map for C (in the sense of [2]). Then there exists a point u in C such that*

$$\| u - f(u) \| = d(f(u), C).$$

Proof. By [2, Theorem 3], p is a nonexpansive map. Therefore $p \circ f : C \to C$ is a continuous 1-set-contractive map and $(p \circ f)(C)$ is bounded, since $f(C)$ is bounded.

It follows by Lemma 1 that $p \circ f$ has a fixed point $u \in C$. Hence

$$\| u-f(u) \| = \| p(f(u))-f\, u) \| = d(f(u), C).$$

The role, which the proximity map plays in Theorem 1, is played in Theorem 2 by the radial retraction : Let $r>0$ and B_r be the closed ball with center at the origin and radius r. Define $R : X \to B_r$ by

$$R(x)=\begin{cases} x & \text{if } \| x \| \leqslant r \\ \dfrac{rx}{\| x \|} & \text{if } \| x \| \geqslant r . \end{cases}$$

Then $\| R(x)-x \| = d(x, B_r)$.

The following theorem generalizes [5, Theorem 1].

Theorem 2. *Assume that $f : B_r \to X$ is a continuous 1-set-contractive map and that $R \circ f$ satisfies (P). Then there exists a point u in B_r such that*

$$\| u-f(u) \| = d(f(u), B_r).$$

Proof. By [7, Proposition 9] R is a continuous 1-set-contractive map, therefore $R \circ f : B_r \to B_r$ is a continuous 1-set-contractive map. It follows from Lemma 1 that $R \circ f$ has a fixed point $u \in B_r$. Hence

$$\| u-f(u) \| = \| R(f(u))-f(u) \| = d(f(u), C).$$

Lin showed in [6, Section 3] that one can obtain various fixed point theorems with the help of his best-approximation theorem [6, Theorem 1]. In these fixed point theorems summarized in [6, Theorem 5], there is always assumed that $(I-p \circ f\,(C)$ is closed.

With the help of our generalization of [6, Theorem 1 of this paper, we can weaken the assumption above to the assumption that $p \circ f$ satisfies the condition (P).

We now give an example of a 1-set-contractive map f, such that f has a fixed point, hence satisfies (P), but the range of $I-f$ is not closed.

Example. Let B be the closed unit ball with center at the origin in the space l_2. Define $T : B \to l_2$ by

$$T(x) = (\| x \|_2, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots) , \ x=(x_n) \in B.$$

Then $f := I-T$ is a 1-set-contractive map, $f(0)=0$ and $e_1=(1, 0, 0, \dots) \in \overline{(I-f)(B)} / (I-f)(B)$.

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Jñānabha, Vol. 20, 1990

Best Approximation and Fixed Point Theorems*

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(Received : May 30, 1990)

The purpose of this paper is to use a fixed point theorem of Edelstein [5] in order to obtain a theorem of best approximation. Further fixed point theorems will be inferred.

Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $a \in X$; then there exists a unique point $P(a) \in C$ closest to a, *i.e.*

$$\| P(a) - a \| = \inf_{x \in C} \| x - a \| =: d(a, C)$$

(*e.g.*, see [3, *p.* 22]). The map $P : X \to C$ is called the proximity map. If X is a Hilbert space then P is a nonexpansive map [4, theorem 3].

Proposition 1. If $f : C \to X$ is a map from C into X and u is a fixed point of $P \circ f$, then

$$\| u - f(u) \| = d(f(u), C).$$

Various authors combine this proposition with known fixed point theorems to obtain criteria for the existence of a point u for which $\| u - f(u) \| = d(f(u), C)$; from this best approximation results they get further fixed point theorems ([1], [2], [6], [7], [8], [9]).

In this note, using the same idea, we combine proposition 1 with the fixed point theorem [5, theorem 2] of of Edelstein to obtain a criterion for the existence of a point $u \in C$ of minimal distance from $f(u)$ for a given map $f : C \to X$ (theorem 1). From this result a fixed point theorem (theorem 3) will be deduced. Our results generalize theorems by Singh and Watson [9, theorem 5 and theorem 6] and by Schoneberg [9, theorem 6].

First we recall the notion of the asymptotic center of a sequence $\{u_n\}$, which is essential in [5]. Let $\{u_n\}$ be a bounded sequence in C. Define

$$r_m(y) := \sup \{ \| u_k - y \| : k \geqslant m \} \quad (y \in X).$$

It is known that there exists a unique point $c_m \in C$ such that

$$r_m(c_m) = \inf \{ r_m (y \in C \}.$$

Edelstein [5, theorem 1] proves that $\{c_m\}$ converges to a point $u \in C$ called the asymptotic center of $\{u_n\}$ with respect to C. The following result due to Edelstein will be used in our theorem.

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* This work was financed by Funds of M. P. I. of Italy.

Theorem 1. (Edelstein [5, theorem 2]). Let $f: C \to C$ be a map from C into C and let $x \in C$ be such that the sequence defined by $u_n = f^n(x)$ is bounded. Denote by u the asymptotic center of $\{u_n\}$ with respect to C. Suppose that there exists a positive integer n_0 and neighborhood V of u in C such that

$$\| f^k(x) - f(v) \| \leqslant \| f^{k-1}(x) - v \| \quad (k \geqslant n_0,\ v \in V).$$

Then $$f(u) = u.$$

Now from proposition 1 and theorem 1 it follows immediately :

Theorem 2. Let $f: C \to X$ be a map from C into X and let $x \in C$ be such that the cequence defined by $u_n := (P \circ f)^n(x)$ is bounded. Denote by u the asymptotic center of $\{u_n\}$ with respect to C. Suppose that there exists a positive integer n_0 and a neighbourhood V of u such that

$$\| (P \circ f)^k(x) - (P \circ f)(v) \| \leqslant \| (P \circ f)^{k-1}(x) - v \|$$

for all integers $k \geqslant n_0$ and all $v \in V \cap C$. Then u is a fixed point of $P \circ f$ and $\| u - f(u) \| = d(f(u), C)$.

Theorem 3. Let X be a Hilbert space and $f: C \to X$ a continuous map such that $f(C)$ is bounded. Suppose that for every $x \in C$ there is a positive integer $n = n(x)$ such that

$$\| P \circ f)^k(x) - (P \circ f)^k(y) \| \leqslant \| (P \circ f)^{k-1}(x) - (P \circ f)^{k-1}(y) \|$$

for all integers $k \geqslant n$ and all $y \in C$. Then there exists a fixed point u of $P \circ f$ and $\| u - f(u) \| = d(f(u), C)$.

Proof. Since the map P is nonexpansive, the closed convex hull $B := \overline{co}\,(P \circ f)(C)$ is bounded and $P \circ f$ is continuous. Therefore, by [5, corollary] the map $P \circ f_{|B} : B \to B$ has a fixed point $u \in B$. Hence one obtaine as in proposition 1

$$\| u - f(u) \| = \| P(f(u) -) f(u) \| = d(f(u), C).$$

If f is nonexpansive and X is a Hilbert space, then $(P \circ f)^k$ is nonexpansive for every positive integer k. Therefore, from theorem 3, one gets the following of Singh and Watson.

Corollary 1. (Singh and Watson [9, theorem 5]). Let X be a Hilbert space and $f: C \to X$ be a nonexpansive map such that $f(C)$ is bounded. Then there exists a point $u \in C$ such that

$$\| u - f(u) \| = d(f(u), C).$$

Remark. It can happen that $P \circ f$ is nonexpansive while f is not : take, *e.g.* $f: [0, 1] \to R$, $f(x) = 2x + 1$. Here the assumptions of theorems 1 and 2 are satisfied, but not those of corollary 1.

Until now we have studied conditions for the existence of a point $u \in C$ with $\| u - f(u) \| = d(f(u), C)$. We will now study how to get fixed point theorems from such results. The following lemma formalizes the method just used in the proof of [9, theorm 6].

Lemma. Let X be a Hilbert space and $f : C \to X$ a map; let u, $y \in C$ be such that u is a fixed point of $P \circ f$ and $\| f(u)-y \| \leqslant \| u-y \|$, Then u is a fixed point of f.

Proof. The segment $S := \{y+t(u-y) : t \in [0, 1]\}$ is contained in C and from $\| u-f(u) \| = \| (P \circ f)(u)-f(u) \| = d(f(u), C)$, it follows that $d(f(u), S) \leqslant \| u-f(u) \| = d(f(u), C) \leqslant d(f(u), S)$, *i. e.*

$$\| u-f(u \| = d((f(u), S). \qquad \text{...(1)}$$

The segment S joins the center y of B and the point u of the boundary ∂B. But, since X is a Hilbert space, for every point $z \in B / \{u\}$ we have $\| z-u \| \neq d(z,S)$, and so thanks to (1), we conclude $u=f(u)$.

Theorem 4. Let X be a Hilbert space and suppose that the assumptions either in theorem 2 or in theorem 3 or in corollary 1 are satisfied. Assume further that there is an element $y \in C$ such that $\| f(x)-y \| \leqslant \| x-y \|$ for every $x \in \partial C$. Then f has a fixed point.

Proof. Our hypotheses imply that the map $P \circ f$ has a fixed point $u \in C$ and $\| u-f(u) \| = d(f(u), C)$. Suppose that $f(u) \notin C$. Then $u \in \partial C$, hence $\| f(u)-y \| \leqslant \| u-y \|$ and, by the lemma, $f(u)=u \in C$. If $f(u) \in C$, then $f(u)=(P \circ f)(u)=u$.

Theorem 4 contains the following result by Schöneberg.

Corollary 2. (Singh and Watson [9, theorem 6]). Let X be Hilbert space and $f : C \to X$ a nonexpansive map such that $f(C)$ is bounded. Assume further that there is an element $y \in C$ such that $\| f(x)-y \| \leqslant \| x-y \|$ for every $x \in \partial C$. Then f has a fixed point.

Proposition 2. Let B_r $(r>0)$ be the closed ball in X with center at the origin and radius r. Let $f : B_r \to X$ be a map with the property that

"if $f(x)=ax$ for some x on the boundary of B_r then $a \leqslant 1$."

Then, if u is a fixed point of $P \circ f$, u is also a fixed point of f.

Proof. By proposition 1 we have $\| u-f(u) \| = d(f(u), B_r)$. Therefore it is enough to prove that $f(u) \in B_r$. Suppose that $f(u) \notin B_r$. Then u is on the boundary of B_r and $f(u)=ax$ for some $a>0$. By assumption $a \leqslant 1$ and therefore $f(u)=au \in B_r$.

From proposition 2 one can obtain fixed point theormes under conditions which imply a fixed point for $P \circ f$, *e.g.* under the assumption either theorem 2 or theorem 3 or corollary 1. If one assumes as in corollary 1 that X is a Hilbert space and f is a nonexpansive map, then one obtains from proposition 2 the theorem by Singh and Watson [9, theorem 8].

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Jñānabha; Vol. 20, 1990

Fractional Integration and Integral Representations of Karlsson's Multiple Hypergeometric Function and its Confluent Forms

by

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(Received : January 31, 1989)

1. INTRODUCTION

Joshi [6] made use of the theory of fractional integration in order to deduce Eulerian Integral representations of hypergeometric functions of three variables, viz. H_A, H_B and H_C defined and studied by Srivastava (*cf* [10] to [14] see also [7] and [15]). In subsequent paper, Chandel [1], obtained simllar integral reprecentations for the multiple hypergeometric functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ of Lauricella [9] and also for their confluent forms $\Phi_2{}^{(n)}$ and $\psi_2{}^{(n)}$. Further in another paper. Chandel [2] deduced similar integral representations for Exton's multiple hypergeometric functions ${}^{(k)}_{(1)}E_D^{(n)}$, ${}^{(k)}_{(2)}E_D^{(n)}$ [5] related to Lauricella's $F_D^{(n)}$ and also for his own multiple hypergeometric function ${}^{(k)}_{(1)}E_C^{(n)}$ related to Lauricella's $F_C^{(n}$. Recently, Chandel and Gupta [3] have defined and studied multiple hypergeometric functions ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$, ${}^{(k)}F_{BD}^{(n)}$ related to Lauricella's functions [9]. Motivated by this paper karlsson [8], also introduced fourth possible intermediate Lauricella function ${}^{(k)}F_{CD}^{(n)}$. Very recently the author Chandel and Vishwakarma [4] have studied confluent forms ${}^{(k)}_{(2)}\Phi_{CD}^{(n)}$, ${}^{(k)}_{(3)}\Phi_{CD}^{(n)}$ ${}^{(k)}_{(4)}\Phi_{CD}^{(n)}$ of ${}^{(k)}F_{CD}^{(n)}$.

In the present paper, we shall make use of the theory of fractional integration to derive Eulerian integral representation for multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$ of Karlsson [8] and for its confluent forms ${}^{(k)}_{(2)}\Phi_{CD}^{(n)}$,

${}^{(k)}_{(3)}\Phi^{(n)}_{CD}$ and ${}^{(k)}_{(4)}\Phi^{(n)}_{CD}$ introduced by the authors [4].

We recall that the rule for fractional integration by parts is given by

$$(1.1) \qquad \int_a^b U \frac{\partial^v V}{\partial(b-x,^{v}}\, dx = \int_a^b V \frac{\partial^v U}{\partial(x-a)^v}\, dx.$$

If Re $(\nu) > 0$, the fractional derivatives occuring in (1.1) are defined by the following integrals :

$$(1.2) \qquad \frac{\partial^\nu U}{\partial(x-a)^v} = \frac{1}{\Gamma(-v)} \int_a^x (x-y)^{-v-1}\ U(y)\ dy.$$

$$(1.3) \qquad \frac{\partial^v V}{\partial(b-x)^v} = \frac{1}{\Gamma(-v)} \int_x^b (y-x)^{-v-1}\ V(v)\ dy.$$

If Re $(\nu) > 0$ and U, V are expressible in terms of the series

$$U = \Sigma\ A_r(x-a)^{\rho+r-1},\ V = \Sigma\ B_s(b-x)^{\rho+s-1}$$

then the fractional derivatives are obtained by differentiating the above series term by term, with the help of the formula.

$$(1.4) \qquad \frac{\partial^v W^{\mu-1}}{\partial W^v} = \frac{\Gamma(\mu)}{\Gamma(\mu-\nu)}\, W^{\mu-v-1}$$

provided $\mu \neq \nu$.

2. THE INTEGRAL REPRESENTATIONS

In this section we derive the integral representations for the functions ${}^{(k)}F^{(n)}_{CD}$, ${}^{(k)}_{(2)}\Phi^{(n)}_{CD}$, ${}^{(k)}_{(3)}\Phi^{(n)}_{CD}$ and ${}^{(k)}_{(4)}\Phi^{(n)}_{CD}$.

Consider

$$\frac{\partial^{\beta_1+\nu_1+\ldots+\beta_k-\nu_k}}{\partial x_1^{\beta_1-\nu_1}\ldots\partial x_k^{\beta_k-\nu_k}} \Big\{ x_1^{\beta_1-1} \ldots x_k^{\beta_k-1}$$

$$ {}^{(k)}F^{(n)}_{CD}\,[a, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n] \Big\}$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\ldots m_n}\,(\nu)_{m_{k+1}+\ldots+m_n}\,(\nu_n)_{m_1}\cdots(\nu_k)_{m_k}}{(\gamma)_{m_1+\ldots+m_k}\,(\gamma_{k+1})_{m_{k+1}}\ldots(\gamma_n)_{m_n}} \cdot \frac{1}{m_1!\ldots m_n!}$$

$$\frac{\partial^{\beta_1-\nu_1+\ldots+\beta_k-\nu_k}}{\partial x_1^{\beta_1-\nu_1}\ldots\partial x_k^{\beta_k-\nu_k}} \Big\{ x_1^{\beta_1+m_1-1} \ldots x_k^{\beta_k-m_k-1}$$

$$x_{k+1}^{m_{k+1}} \ldots x_n^{m_n} \Big\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\beta_j)}{\Gamma(\nu_j)} (x_j)^{\nu_j - 1}\; {}^{(k)}F_{CD}^{(n)} [a, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n;\ x_1, \ldots, x_n]$$

Hence by using the relation (1.2), we have

$$ {}^{(k)}F_{CD}^{(n)} [a, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)} (x_j)^{1-\nu_j} \quad \frac{\partial^{\beta_1-\nu_1+\ldots+\beta_k-\nu_k}}{\partial x_1^{\beta_1-\nu_1} \ldots \partial x_k^{\beta_k-\nu_k}} \Big\{ x_1^{\beta_1-1} \ldots x_k^{\beta_k-1}$$

$${}^{(k)}F_{CD}^{(n)} [a, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n] \Big\}$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)(x_j)^{-\nu_j+1}}{\Gamma(\beta_j)\ \Gamma(\nu_j-\beta_j)} \int_0^{x_1} \ldots \int_0^{x_k} (x_1-y_1)^{\nu_1-\beta_1-1}\, y_1^{\beta_1-1}$$

$$\ldots (x_k-y_k)^{\nu_k-\beta_k-1}\, y_k^{\beta_k-1}$$

$${}^{(k)}F_{CD}^{(n)} [a, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; y_1, \ldots, y_k, x_{k+1},$$

$$\ldots, x_n]\, dy_1 \ldots dy_k.$$

Now putting every $y_j = x_j t_j$, where $j = 1, \ldots, k$, we establish

(2 1) $${}^{(k)}F_{CD}^{(n)} [a, \nu, \beta_1, \ldots, \beta_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= \prod_{j=1}^{k} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\ \Gamma(\nu_j-\beta_j)} \int_0^1 \ldots \int_0^1 \prod_{i=1}^{k} (1-t_i)^{\nu_i-\beta_i-1}\ (t_i)^{\beta_i-1}$$

$${}^{(k)}F_{CD}^{(n)} [a, \nu, \nu_1, \ldots, \nu_k; \gamma, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, \ldots, x_k t_k, x_{k+1}, \ldots, x_n]$$

$$dt_1 \ldots dt_k,$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, $\quad i = 1, \ldots, k.$

For brevity we use the operator Ω defined by chandel [1, (2.2)] :

(2.2) $$\Omega^{\beta_1, \ldots, \beta_k}_{\nu_1, \ldots, \nu_k} \{ \quad \}$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)}\int_0^1\cdots\int_0^1\prod_{i=1}^{k}(t_i)^{\beta_i-1}\,(1-t_i)^{\nu_i-\beta_i-1}\;\{\}\,dt_1\ldots dt_k$$

where $\qquad 0<\mathrm{Re}\,(\beta_i<\mathrm{Re}\,(\nu_i),\ i=1,\ldots,k.$

Hence the relation (2.1) can be written as

(2 3) $\quad \Omega\left\{{}^{(k)}F_{CD}^{(n)}[a,\nu,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n]\right\}$

$$={}^{(k)}F_{CD}^{(n)}[a,\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1\ldots,x_n]$$

provided $\qquad 0<\mathrm{Re}\,(\beta_i)<\mathrm{Re}\,(\nu_i),\ i=1,\ldots,k.$

Applying the same techniques, we also obtain the following operational relations.

(2.4) $\quad \Omega\left\{{}^{(k)}_{(2)}\Phi_{CD}^{(n)}[a,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots x_n]\right\}$

$$={}^{(k)}_{(2)}\Phi_{CD}^{(n)}[a,\beta_1,\ldots,\beta_k\ \gamma,\gamma_{k+1},\ldots,\gamma_n;x_1\ldots,x_n]$$

provided $\qquad 0<\mathrm{Re}\,(\beta_i)<\mathrm{Re}\,(\nu_i),$ where $i=1,\ldots,k.$

(2.5) $\quad \Omega\left\{{}^{(k)}_{(3)}\Phi_{CD}^{(n)}[\nu,\nu_1,\ldots,\nu_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1t_1\ldots,x_kt_k,x_{k+1},\ldots,x_n]\right\}$

$$={}^{(k)}_{(3)}\Phi_{CD}^{(n)}[\nu,\beta_1,\ldots,\beta_k;\gamma,\gamma_{k+1},\ldots,\gamma_n;x_1,\ldots,x_n],$$

provided $\qquad 0<\mathrm{Re}\,(\beta_i)<\mathrm{Re}\,(\nu_i),$ where $i=1,\ldots,k.$

(2.6) $\quad \Omega\left\{{}^{(k)}_{(4)}\Phi_{CD}^{(n)}[a,\nu,\nu_1,\ldots,\nu_k;\gamma;x_1t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n]\right\}$

$$={}^{(k)}_{(4)}\Phi_{CD}^{(n)}[a,\nu,\beta_1,\ldots,\beta_k;\gamma;x_1,\ldots,x_n]$$

provided $\qquad 0<\mathrm{Re}\,(\beta_i)<\mathrm{Re}\,(\nu_i),\ i=1,\ldots,k.$

(2.7) $\quad \Omega\Big\{{}^{(k)}F_{CD}^{(n)}[a,\nu,\gamma_1,\nu_2,\ldots,\nu_k;\beta_1,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,x_1x_2t_1t_2,\ldots,$

$$x_1x_kt_1t_k,x_{k+1},\ldots,x_n]\Big\}$$

$$={}^{(k)}F_{CD}^{(n)}[a,\nu,\gamma_1,\beta_2,\ldots,\beta_k;\nu_1,\gamma_{k+1},\ldots,\gamma_n;x_1,x_1x_2,\ldots,$$

$$x_1x_k,x_{k+1},\ \ ,x_n],$$

provided $\qquad 0<\mathrm{Re}\,(\beta_i)<\mathrm{Re}\,(\nu_i)=1,\ldots,k.$

(2.8) $\quad \Omega\Big\{{}^{(k)}_{(2)}\Phi_{CD}^{(n)}[a,\gamma_1\ \nu_2,\ldots,\nu_k;\beta_1,\gamma_{k+1},\ldots,\gamma_n;x_1t_1,x_1x_2t_1t_2,\ldots,x_1x_kt_1t_k,$

$$x_{k+1},\ldots,x_n]\Big\}$$

$$= {}_{(2)}^{(k)}\Phi_{CD}^{(n)}[\alpha, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1} \ldots, \gamma_n; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n]$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i),\ i = 1, \ldots, k.$

$$(2.9)\quad \Omega\left\{{}_{(3)}^{(k)}\Phi_{CD}^{(n)}[\nu, \gamma_1, \nu_2, \ldots, \nu_k; \beta_1, \gamma_{k+1}, \ldots, \gamma_n; x_1 t_1, x_1 x_2 t_1 t_2, \ldots, x_1 x_k t_1 t_k, x_{k+1}, \ldots, x_n]\right\}$$

$$= {}_{(3)}^{(k)}\Phi_{CD}^{(n)}[\nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n],$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i),\ i = 1, \ldots, k.$

$$(2.10)\quad \Omega\left\{{}_{(4)}^{(k)}\Phi_{CD}^{(n)}[\alpha, \nu, \gamma_1, \nu_2, \ldots, \nu_k; \beta_1; x_1 t_1, x_1 t_1 x_2 t_2, \ldots, x_1 t_1 x_k t_k, x_{k+1}, \ldots, x_n]\right\}$$

$$= {}_{(4)}^{(k)}\Phi_{CD}^{(n)}[\alpha, \nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1; x_1, x_1 x_2, \ldots, x_1 x_k, x_{k+1}, \ldots, x_n],$$

provided $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i),\ i = 1, \ldots, k.$

3. FURTHER EXTENSIONS

In this section, we derive extensions of the results obtained in the preceding section to hold for multiple hypergeometric function, defined by

$$\overset{n}{F}(x_1, \ldots, x_n) = F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & (-;\nu) & \\ k & \lambda_1, \nu_1; \ldots; \lambda_k, \nu_k, & x_1, \ldots, x_n \\ 0 & \ldots \quad \ldots \quad \ldots & \\ 1 & (\gamma;-) & \\ n & \mu_1; \ldots, \mu_k; \gamma_{k+1}; \ldots; \gamma_n & \end{array}\right]$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(\alpha)_{m_1 + \ldots + m_n} (\nu)_{m_{k+1} + \ldots + m_n} (\lambda_1)_{m_1} \ldots (\lambda_k)_{m_k} (\nu_1)_{m_1} \ldots (\nu_k)_{m_k}}{(\gamma)_{m_1 + \ldots + m_k} (\mu_1)_{m_1} \ldots (\mu_k)_{m_k} (\gamma_{k+1})_{m_{k+1}} \ldots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}.$$

From the above defintion it follows that

$$\frac{\partial^{\mu_1-\lambda_1+\dots+\mu_k-\lambda_k}}{\partial t_1^{\mu_1-\lambda_1}\dots\partial t_k^{\mu_k-\lambda_k}}\left\{t_1^{\mu_1-1}\dots t_k^{\mu_k-1}\overset{\infty}{F}(x_1t_1,\dots,x_kt_k,x_{k+1},\dots,x_n)\right\}$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\mu_j)}{\Gamma(\lambda_j)}(t_j)^{\lambda_j-1}\;{}^{(k)}F_{CD}^{(n)}[a,\nu,\nu_1,\dots,\nu_k;\gamma,\gamma_{k+1},\dots,\gamma_n;x_1t_1,\dots,x_kt_k,x_{k+1},\dots,x_n].$$

Using the relation (2.3), we have

$${}^{(k)}F_{CD}^{(n)}[a,\nu,\beta_1,\dots,\beta_k;\gamma,\gamma_{k+1},\dots,\gamma_n;x_1,\dots,x_n]$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j-\beta_j)}\int_0^1\dots\int_0^1\prod_{i=1}^{k}(t_i)^{\beta_i-\lambda_i}(1-t_i)^{\nu_i-\beta_i-1}$$

$$\frac{\partial^{\mu_1-\lambda_1+\dots+\mu_k-\lambda_k}}{\partial t_1^{\mu_1-\lambda_1}\dots\partial t_k^{\mu_k-\lambda_k}}\left\{t_1^{\mu_1-1}\dots t_k^{\mu_k-1}\overset{\infty}{F}(x_1t_1,\dots,x_kt_k,x_{k+1},\dots,x_n)\right\}dt_1\dots dt_n.$$

By using the relation (1.1), we further obtain

$${}^{(k)}F_{CD}^{(n)}[a,\nu,\beta_1,\dots,\beta_k;\gamma,\gamma_{k+1},\dots,\gamma_n;x_1,\dots,\lambda_n]$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j-\beta_j)}\int_0^1\dots\int_0^1 t_1^{\mu_1-1}\dots t_k^{\mu_k-1}$$

$$\overset{\infty}{F}(x_1t_1,\dots,x_kt_k\,x_{k+1},\dots,x_n)\;\frac{\partial^{\mu_1-\lambda_1+\dots+\mu_k-\lambda_k}}{\partial(1-t_1)^{\mu_1-\lambda_1}\dots\partial(1-t_k)^{\mu_k-\lambda_k}}$$

$$\left\{\prod_{i=1}^{k}(t_i)^{\beta_i-\lambda_i}(1-t_i)^{\nu_i-\beta_i-1}\right\}dt_1\dots dt_k$$

$$=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j+\lambda_j-\mu_j-\beta_j)}\int_0^1\cdot\int_0^1\prod_{j=1}^{k}(t_i)^{\mu_i-1}$$

$$(1-t_i)^{\mu_i+\lambda_i-\mu_i-\beta_i-1}$$

$${}_2F_1[\nu_i-\beta_i,\lambda_i-\beta_i;\nu_i+\lambda_i-\mu_i-\beta_i;(1-t_i)]\overset{\infty}{F}(x_1t_1,\dots,x_kt_k,x_{k+},\dots,x_n)\,dt_1\dots dt_k.$$

For brevity, we use the following operator **R** introduced by Chandel [1, (3.1)] :

$$(3.1)\qquad R\left\{\quad\right\}=\prod_{j=1}^{k}\frac{\Gamma(\nu_j)\,\Gamma(\lambda_j)}{\Gamma(\beta_j)\,\Gamma(\mu_j)\,\Gamma(\nu_j+\lambda_j-\beta_j-\mu_j)}\int_0^1\cdots\int_0^1\prod_{j=1}^{k}(t_i)^{\mu_i-1}$$

$$(1-t_i)^{\nu_i+\lambda_i-\mu_i-\beta_i-1}\;{}_2F_1\,[\nu_i-\beta_i,\ \lambda_i-\beta_i;\ \nu_i+\lambda_i-\mu_i-\beta_i;\ (1-t_i)]$$

$$\left\{\quad\right\}dt_1\ \dots\ dt_k.$$

We then find the following result

$$(3.2)\qquad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 1 & (-;\ \nu) & \\ \mathrm{k} & \lambda_1,\ \nu_1;\ \dots;\ \lambda_k,\ \nu_k & \\ 0 & \cdots\quad\cdots\quad\cdots & x_1t_1,\ \dots,\ x_kt_k,\ x_{k+1},\ \dots,\ x_n \\ 1 & (\gamma;\ -) & \\ n & \mu_1;\ \dots;\ \mu_k;\ \gamma_{k+1};\ \dots;\ \gamma_n & \end{array}\right]\right\}$$

$$=\overset{(k)\ (n)}{F}_{CD}\ [a,\ \nu,\ \beta_1,\ \dots,\ \beta_k;\ \gamma,\ \gamma_{k+1},\ \dots,\ \gamma_n;\ x_1,\ \dots,\ x_n],$$

provided that $0<\mathrm{Re}\ (\mu_j)<\mathrm{Re}\ (\nu_j+\lambda_j-\beta_j);\ 0<\mathrm{Re}\ (\beta_j)<\mathrm{Re}\ (\nu_j),\ j=1,\ \dots,\ k.$

[Extension of (2.3)]

Applying the same techniques we obtain the following results :

$$(3.3)\qquad R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 0 & \cdots\quad\cdots\quad\cdots & \\ \mathrm{k} & \lambda_1,\ \nu_1;\ \dots;\ \lambda_k,\ \nu_k & \\ 0 & \cdots\quad\cdots\quad\cdots & x_1t_1,\ \dots,\ x_kt_k,\ \lambda_{k+1},\ \dots,\ x_n \\ 1 & (\gamma;\ -) & \\ n & \mu_1;\ \dots;\ \mu_k;\ \gamma_{k+1};\ \dots;\ \gamma_n & \end{array}\right]\right\}$$

$$=\ {}_{(2)}\overset{(k)\ (n)}{\Phi}_{CD}\ [a,\ \beta_1,\ \dots,\ \beta_k;\ \gamma,\ \gamma_{k+1},\ \dots,\ \gamma_n;\ x_1,\ \dots,\ x_n]$$

valid if $0<\mathrm{Re}\ (\beta_i)<\mathrm{Re}\ (\nu_i);\ 0<\mathrm{Re}\ (\mu_i)<\mathrm{Re}\ (\nu_i+\lambda_i-\beta_i),\ i=1,\ \dots,\ k.$

[Extension of (2.4)]

$$(3.4)\qquad R\left\{F\left[\begin{array}{c|c|c} 0 & \cdots\quad\cdots\quad\cdots & \\ 1 & (-;\ \nu) & \\ k & \lambda_1,\ \nu_1;\ \dots;\ \lambda_k,\ \nu_k & \\ 0 & \cdots\quad\cdots\quad\cdots & x_1t_1,\ \dots,\ x_kt_k,\ x_{k+1},\ \dots,\ x_n \\ 1 & (\gamma;\ -) & \\ n & \mu_1;\ \dots;\ \mu_k;\ \gamma_{k+1};\ \dots,\ \gamma_n & \end{array}\right]\right\}$$

$$=\ {}_{(3)}\overset{(k)\ (n)}{\Phi}_{CD}\ [\nu,\ \beta_1,\ \dots,\ \beta_k;\ \gamma,\ \gamma_{k+1},\ \dots,\ \gamma_n;\ x_1,\ \dots,\ x_n],$$

provided that $0<\text{Re}\,(\beta_i)<\text{Re}(\nu_i)$; $0<\text{Re}\,(\mu_i)<\text{Re}\,(\nu_i+\lambda_i-\beta_i)$, where $i=1,\ldots,k$.

[Extension of (2.5)]

$$(3.5)\quad R\left\{F\left[\begin{array}{c|c|l} 1 & a & \\ 1 & (-;\nu) & \\ k & \lambda_1,\nu_1;\ldots;\lambda_k,\nu_k & x_1\,t_1,\ldots,x_kt_k,x_{k+1},\ldots,x_n \\ 0 & \ldots\quad\ldots\quad\ldots & \\ 1 & (\gamma;-) & \\ k & \mu_1;\ldots;\mu_k & \end{array}\right]\right\}$$

$$={}^{(k)}_{(4)}\Phi^{(n)}_{CD}\,[a,\nu,\beta_1,\ldots,\beta_k;\gamma;x_1,\ldots,x_n]$$

valid if $0<\text{Re}\,(\mu_i)<\text{Re}\,(\nu_i+\lambda_i-\beta_i)$; $0<\text{Re}\,(\beta_i)<\text{Re}\,(\nu_i)$, $i=1,\ldots,k$.

[Extension of (2.6)]

$$(3.6)\,R\left\{F\left[\begin{array}{c|c|c} 1 & a & \\ 1 & (\lambda_1;\nu) & \\ k & \gamma_1;\lambda_2,\nu_2;\ldots;\lambda_k\,\nu_k, & x_1t_1,x_1\,x_2t_1t_2,\ldots,x_1x_k \\ 0 & \ldots\quad\ldots\quad\ldots & t_1t_k,x_{k+1},\ldots,x_n \\ 1 & (\nu_1,\mu_1;-) & \\ n\text{-}1 & \mu_2;\ldots;\mu_k;\gamma_k;\gamma_{k+1};\ldots;\gamma_n & \end{array}\right]\right\}$$

$$={}^{(k)}F^{(n)}_{CD}\,[a,\nu,\gamma_1,\beta_1,\ldots,\beta_k;\nu_1,\ \gamma_{k+1},\ldots,\gamma_n;x_1,x_1x_2,\ldots,x_1x_k,x_{k+1},\ldots,x_n],$$

provided that $0<\text{Re}\,(\mu_i)<\text{Re}\,(\nu_i+\lambda_i-\beta_i)$; $0<\text{Re}\,(\beta_i)<\text{Re}\,(\nu_i)$, $i=1,\ \ldots,k$.

[Extension of (2.7)]

$$(3.7)\quad R\left\{F\left[\begin{array}{c|c|r} 1 & a & \\ 1 & (\lambda_1;-) & \\ k & \gamma_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k & x_1t_1,x_1t_1x_1t_2,\ldots,x_1t_2x_kt_k, \\ 0 & \ldots\quad\ldots\quad\ldots & x_{k+1},\ldots,x_n \\ 1 & (\beta_1,\mu_1;-) & \\ n-1 & \mu_2;\ldots;\mu_k;\gamma_{k+1};\ldots;\gamma_n & \end{array}\right]\right\}$$

$$={}^{(k)}_{(2)}\Phi^{(n)}_{CD}\,[a,\gamma_1,\beta_2,\ldots,\beta_k;\nu_1,\gamma_{k+1},\ldots,\gamma_n;x_1,x_1x_2,\ldots,x_1x_k,x_{k+1},\ldots,x_n],$$

provided that $0<\text{Re}\,(\mu_i)<\text{Re}\,(\nu_i+\lambda_i-\beta_i)$; $0<\text{Re}\,(\beta_i)<\text{Re}\,(\nu_i)$, $i=1,\ \ldots,k$.

[Extension of (2.8)]

$$(3.8)\quad R\left\{F\left[\begin{array}{c|c|r} 0 & \ldots\quad\ldots\quad\ldots & \\ 1 & (\lambda_1;\nu) & \\ k & \gamma_1;\nu_2,\lambda_2;\ldots;\nu_k,\lambda_k & x_1t_1,x_1t_1x_2t_2,\ldots,x_1t_1x_kt_k \\ 0 & \ldots\quad\ldots\quad\ldots & x_{k+1}.\ldots,x_n \\ 1 & (\beta_1,\mu_1;-) & \\ n-1 & \mu_2;\ldots;\mu_k;\gamma_{k+1};\ldots,\gamma_n & \end{array}\right]\right\}$$

$$= {}_{(3)}^{(k)}\Phi_{CD}^{(n)}\,[\nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1, \gamma_{k+1}, \ldots, \gamma_n; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1} \ldots, x_n],$$

provided that $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}(\nu_i)$; $0 < \mathrm{Re}\,(\mu_i) < \mathrm{Re}\,(\nu_i + \lambda_i - \beta_i)$, $i = 1, \ldots, k$.

[Extension of (2.9)]

$$\text{(3.9)}\quad R\left\{F\left[\begin{array}{c|c|c} 1 & \alpha & \\ 1 & (\lambda_1; \nu) & \\ k & \gamma_1; \nu_2, \lambda_2; \ldots; \nu_k, \lambda_k & x_1t_1, x_1t_1x_2t_2, \ldots, x_1t_1x_kt_k, \\ 0 & \ldots \quad \ldots \quad \ldots & x_{k+1}, \ldots, x_n \\ 1 & (\beta_1, \mu_1; -) & \\ k-1 & \mu_2; \ldots; \mu_k & \end{array}\right]\right\}$$

$$= {}_{(4)}^{(k)}\Phi_{CD}^{(n)}\,[\alpha, \nu, \gamma_1, \beta_2, \ldots, \beta_k; \nu_1; x_1, x_1x_2, \ldots, x_1x_k, x_{k+1}, \ldots, x_n],$$

provided that $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}(\nu_i)$; $0 < \mathrm{Re}(\mu_i) < \mathrm{Re}(\nu_i + \lambda_i - \beta_i)$, $i = 1, \ldots k$.

[Extension of (2.10)]

ACKNOWLEDGEMENT

The authors are thankful to the Council of Science and Technology, U. P., India for providing financial assistance to this project.

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Jñānābha, Vol. 20, 1990

A Generalized Hankel Transformation of Distributions and a Cauchy Problem Involving the Bessel Type Operator

$$B_{\alpha_0,\,\alpha_1,\,\alpha_2}=x^{\alpha_0}\;Dx^{\alpha_1}\;Dx^{\alpha_2}$$

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(Received : July 15, 1988, Revised : January 12, 1990)

ABSTRACT

In this paper we introduce new spaces of testing functions denoted by $H_{\alpha_2,\,k;\,a,\,A}$, $H^{\beta,\,B}_{\alpha_2,\,k}$ and $H^{\beta,\,B}_{\alpha_2,\,k;\,a,\,A}$ and their topological properties are discussed. These spaces are generalizations of testing functions spaces of type H_μ. It is shown that the $F_{\alpha_0,\,\alpha_1,\,\alpha_2}$ –transformation (a new integral transformation depending on three parameters $(\alpha_0,\alpha_1,\alpha_2)$ that reduces to the ordinary Hankel transformation when $\alpha_0=-\mu-\frac{1}{2}$, $\alpha_1=2\mu+1$ and $\alpha_2=-\mu-\frac{1}{2}$) is a continuous linear mapping from each of the spaces of type $H^{\beta}_{\alpha_2,\,k;\,a}$ into certain other spaces of type $H^{\beta}_{\alpha_2,\,k;\,a}$, when $k=-\alpha_0-\alpha_1-\alpha_2$. The theory thus developed is applied to solve Cauchy problems involving the Bessel type operator $B_{\alpha_0,\,\alpha_1,\,\alpha_2}=x^{\alpha_0}\;Dx^{\alpha_1}\;Dx^{\alpha_2}$.

1. INTRODUCTION

In a previous paper [2] we introduced a generalization of the well known Hankel transformation defined by

$$F(y)=F_{\alpha_0,\,\alpha_1,\,\alpha_2}\{f(x)\}\,(y)=y^{\alpha_0-\alpha_2}\int_0^\infty J(xy;\,\alpha_2,\,\alpha_1,\,\alpha_0)\,f(x)\,dx \quad (1)$$

where $J(z;\,\alpha_0,\,\alpha_1,\,\alpha_2)=z^{(1-\alpha_1-2\alpha_2)/2}\;J_\nu\!\left(\frac{2}{2+k}\;z^{(2+k)/2}\right)$,

$k=-\alpha_0-\alpha_1-\alpha_2$, $\nu=\dfrac{\alpha_1-1}{2+k}\geqslant-\frac{1}{2}$ and J_ν denotes the Bessel function of the first kind and order ν. In the special case when $\alpha_0=\alpha_2=-\mu-\frac{1}{2}$ and

$a_1=2\mu+1$, (1) yields the known Hankel transform. By specializing the parameters we also abtain other variants of the Hankel transform, for example $F_{a_0,\, a_1,\, a_2}$ reduces to the Hankel-Schwartz transformation $B\mu$ for $a_1=-a_0+2\mu+1$ and $a_2=0$ (A. L. Schwartz [12]) and to the Hankel-Clifford transform $ch\mu$ when $a_0=0$, $a_1=\mu+1$ and $a_2=-\mu$ (J. M. Mendez [8]), amongst others.

In [2] the following statements concerning the $F_{a_0,\, a_1,\, a_2}$ transformation were established.

Theorem 1 (Inversion formula) : If f is of bounded variation into a neigbourhood of the point $x_0>0$, $\frac{a_1-1}{2+k}\geqslant-\frac{1}{2}$, $2+k>0$ and the integral

$$\int_0^\infty |f(x)|\, x^{\frac{1-a_1-2a_0}{2}-\frac{2+k}{4}}\, dx \text{ exists, then}$$

$$\lim_{\lambda\to\infty} x_0^{a_0-a_2}\int_0^\infty f(x)\int_0^\lambda J(xu;\, a_2,\, a_1,\, a_0)\, J(x_0 u;\, a_2,\, a_1,\, a_0)\, u^{a_0-a_2}\, du\, dx$$

$$=\tfrac{1}{2}\{f(x_0+0)+f(x_0-0)\}.$$

Theorem 2 (Mixed Parseval equation). If $f(x)x^{-a_0}$ and $G(y)y^{-a_2}$ are in $L_1(0,\infty)$, $F(y)=F_{a_0,\, a_1,\, a_2}\{f(x)\}(y)$ and $G(y)=F_{a_2,\, a_1,\, a_0}\{g(x)\}(y)$ then

$$\int_0^\infty f(x)g(x)dx=\int_0^\infty F(y)\, G(y)\, dy \qquad \ldots(2)$$

provided that $\frac{a_1-1}{2+k}\geqslant-\frac{1}{2}$ and $2+k>0$.

The $F_{a_0,\, a_1,\, a_2}$ —transformation was extended ([2] and [3]) to certain spaces of generalized functions.

In [3] a testing function space $H_{a_2,\, k}$ is constructed, The space consists of all infinitely differentiable function $\phi(x)$ defined on I. I denoting the real interval $0<x<\infty$, such that

$$\gamma_{a_2,\, k}^{m,\, n}(\phi)=\sup_{x\in I}\left| x^{m(2+k)/2}\left(x^{-1-k} D_x\right)^n \left(x^{a_2}\phi(x)\right)\right| <\infty.$$

for every $m, n\in N$. $H_{a_2,k}$ is endowed with the topology generated by the countable multinorm $\left\{\gamma_{a_2,\, k}^{m,\, n}\right\}_{m,\, n\in N}$. We proved that the $F_{a_0,\, a_1,\, a_2}$ —transformation is an automorphism onto $H_{a_2,\, k}$ provided that $2+k>0$,

$\frac{\alpha_1-1}{2+k} \geqslant -\frac{1}{2}$ and $\alpha_0+\alpha_2<1$.

The generalized transformation $F'_{\alpha_0,\alpha_1,\alpha_2}$ was defined on $H'_{\alpha_0,k}$ the dual space of $H_{\alpha_0,k}$, as the adjoint of $F_{\alpha_2,\alpha_1,\alpha_0}$ on $H_{\alpha_0,k}$ by

$$<F'_{\alpha_0,\alpha_1,\alpha_2} f, \phi> = <f, F_{\alpha_2,\alpha_1,\alpha_0}\phi>, \; f\varepsilon H'_{\alpha_0,k} \text{ and } \phi\varepsilon H_{\alpha_0,k} \quad ...(3)$$

$F'_{\alpha_0,\alpha_1,\alpha_2}$ is an automorphism onto $H'_{\alpha_0,k}$, provided that $2+k>0$, $\frac{\alpha_1-1}{2+k} \geqslant -\frac{1}{2}$ and $\alpha_0+\alpha_2<1$.

Note that the definition (3) appears to be a generalization of the mixed Parseval's equation (2).

I. M. Gelfand and G. E. Shilov [4] introduced spaces of type S and used them to solve Cauchy problem. Later W. Y. Lee [7] defined spaces of type H_μ and used them for the study of a Dirichlet problem in cylindrical coordinates.

In this paper, motivated by the works of I. M. Gelfand and G. E. Shilov [4] and W.Y. Lee [7], we introduce spaces of type $H_{\alpha_2,k}$. These new spaces will be denoted generically by $H^{\beta}_{\alpha_2,k;\alpha}$, where α and β are positive real numbers. After studying several topological properties of the spaces $H^{\beta}_{\alpha_2,k;\alpha}$ we define the generalized $F'_{\alpha_0,\alpha_1,\alpha_2}$ —transformation on said spaces. Finally we apply our results to solve Cauchy problems involving the Bessel type operator $B_{\alpha_0,\alpha_1,\alpha_2}=x^{\alpha_0}Dx^{\alpha_1}Dx^{\alpha_2}$.

Some of the results presented herein reduces to the ones due to W. Y. Lee [7] when $\alpha_0=\alpha_2=-\mu-\frac{1}{2}$ and $\alpha_1=2\mu+1$.

Throughout this paper $D(I)$ denotes the space of smooth functions having compact support on I. We equip $D(I)$ with the usual topology. Its dual $D'(I)$ is the space of Schwartz distributions on I [12].

$E(I)$ denotes the space of smooth functions on I. Its dual $E'(I)$ is the space of distributions with compact support.

We will use a few times the following formulas involving the function $J(x;\alpha_0,\alpha_1,\alpha_2)$ (see [2]).

$$\frac{d}{dx}\left(x^{\alpha_2} J(x;\alpha_0,\alpha_1,\alpha_2)\right) = -x^{\alpha_2+k/2} J\left(x;\alpha_0-\frac{2+k}{2}, \alpha_1+2+k, \alpha_2-\frac{2+k}{2}\right) \quad ...(4)$$

$$\frac{d}{dx}\left(x^{\alpha_1+\alpha_2-1}\,J(x;\,\alpha_0,\,\alpha_1,\,\alpha_2)\right) \qquad \ldots(5)$$

$$= x^{-\alpha_0-(2+k)/2}\;J\left(x;\,\alpha_0+\frac{2+k}{2},\,\alpha_1-2+k,\,\alpha_2+\frac{2+k}{2}\right)$$

and the asymptotic behaviours ([2]):

$$J(x;\,\alpha_0,\,\alpha_1,\,\alpha_2)\,x^{(\alpha_1+2\alpha_2-1)/2} \cong \sqrt{\frac{2+k}{\pi}}\;x^{-(2+k)/4}\,.$$

$$\cdot\cos\left(\frac{2}{2+k}\;x^{(2+k)/2}\;-\frac{\pi}{2}\left(\frac{\alpha_1-1}{2+k}-\tfrac{1}{2}\right)\right)$$

$$+\,O(x^{-3(2+k)/4}),\ \text{as}\ x\to 0^- \qquad \ldots(6)$$

and

$$J(x;\,\alpha_0,\,\alpha_1,\,\alpha_2)\cong\frac{(2+k)^{(1-\alpha_1)/(2+k)}}{\Gamma\left(1+\dfrac{\alpha_1-1}{2+k}\right)}\;x^{-\alpha_2}\ ,\ \text{as}\ x\to 0^+ \qquad \ldots(7)$$

2. SPACES OF TYPE $H_{\alpha_2,\,k}$ $H_{\alpha_2,\,k;\,\alpha}$, $H^{\beta}_{\alpha_2,\,k}$ AND $H^{\beta}_{\alpha_2,\,k;\,\alpha}$

In this section we introduce spaces of type $H_{\alpha_2,\,k}$ denoted by $H_{\alpha_2,k;\,\alpha}$ $H^{\beta}_{\alpha_2,\,k}$ and $H^{\beta}_{\alpha_2,\,k;\,\alpha}$ and we study several of its topological properties.

2.1 The space $H_{\alpha_2,\,k;\,\alpha}$.

Let α_2, $k\in R$, $\alpha\geqslant 0$ and $A>0$. We define $H_{\alpha_2,\,k;\,\alpha,\,A}$ as the space of testing functions $\phi(x)$ which are defined and smooth on $0<x<\infty$ and for which

$$\sup_{x\in I}\left|x^{m(2+k)/2}\;(x^{-1-k}\;D)^q\;(x^{\alpha_2}\,\phi(x))\right|\leqslant C_{q,\,\delta}\,(A+\delta)^m\,m^{m\alpha}$$

for every m, $q\in N$ and $\delta>0$. $C_{q,\,\delta}$ is a constant depending upon the function ϕ. For $m=0$, we set $m^{m\alpha}=1$.

We assign to $H_{\alpha_2,\,k;\,\alpha,\,A}$ the topology generated by the multinorm $\Gamma_1=\{\|\ \ \|_{\delta,\,q}\}_{\delta>0,\,q\in N}$, where

$$\|\phi\|_{\delta,\,q}=\sup_{x\in I,\ m\in N}\frac{\left|x^{m(2+k)/2}\,(x^{-1-k}D)^q\,(x^{\alpha_2}\,\phi(x))\right|}{(A+\delta)^m\,m^{m\alpha}},$$

for ever $\phi \in H_{\alpha_2, k;\, a,\, A}$. It is easily seen that Γ_1 is equivalent to the countable multinorm $\Gamma_2 = \{\| \ \|_{1/n,\, q}\}_{n,\, q \in N}$. Hence, $H_{\alpha_2, k;\, a,\, A}$ is a Hausdorff locally convex linear space that satisfies the first axiom of countability. The dual $H'_{\alpha_2, k;\, a,\, A}$ is a linear space endowed with the weak topology.

We now list some properties of these spaces.

Proposition 1 : $H_{\alpha_2, k;\, a,\, A} \subset H_{\alpha_2, k}$ and the inclusion is continuous.

To see this assertion it is sufficient to note that

$$\sup_{x \in I} \left| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) \right| \leqslant$$

$$\leqslant (A+\delta)^m m^{m\alpha} \sup_{x \in I,\ m \in N} \frac{\left| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) \right|}{(A+\delta)^m m^{m\alpha}}$$

for every q, $m \in N$, $\delta > 0$ and $\phi \in H_{\alpha_2, k;\, a,\, A}$.

Hence, in virtue of charcterization theorem ([2], *pp.* 73), if ϕ is in $H_{\alpha_2, k;\, a,\, A}$ then ϕ satisfies the following three properties

(*i*) $\phi \in C^\infty(I)$,

(*ii*) $D^q \phi(x)$ is of rapid descent as $x \to \infty$ for every $q \in N$,

(*iii*) $\phi(x) = x^{-\alpha_2} (a_0 + a_1 x^{2+k} + \dots + a_n x^{n(2+k)} + o(x^{n(2+k)}))$, for all $n \in N$, as $x \to 0^+$.

Moreover, since $H_{\alpha_2, k}$ is contained in $E(I)$, $H_{\alpha_2, k;\, a,\, A}$ is also contained in $E(I)$.

Proposition 2 : $H_{\alpha_2, k;\, a,\, A}$ is a Frechet space.

Proof. We have to prove that if $\left\{\phi_v\right\}_{v=0}^{\infty}$ is an arbitrary Cauchy sequence in $H_{\alpha_2, k;\, a,\, A}$ then $\left\{\phi_v\right\}_{v=0}^{\infty}$ is a convergent sequence in $H_{\alpha_2, k;\, a,\, A}$. Indeed, since $H_{\alpha_2, k;\, a,\, A}$ is contained in $H_{\alpha_2, k}$, $\left\{\phi_v\right\}_{v=0}^{\infty}$ is a Cauchy sequence in $H_{\alpha_2, k}$. Hence there exists a function $\phi \in H_{\alpha_2, k}$ such that $\phi_v \to \phi$, as $v \to \infty$ in $H_{\alpha_2, k}$. Then, for $\varepsilon > 0$ and $m, q \in N$ there exists $v_0 \in N$ such that for every $v > v_0$

$(A+\delta)^{-m} m^{-m\alpha} \mid x^{m(2+k)/2} (x^{-1-k}D)^q (x^{a_2}(\phi(x)-\phi_v(x))\mid < \varepsilon$, for every $x \epsilon (0, \infty)$.

Therefore $\phi_v \to \phi$ as $v \to \infty$ in $H_{a_2, k; a, A}$.

Proposition 3. Let $\alpha > 0$. $D(I) \subset H_{a_2, k; a, A}$ and the topology of $D(I)$ is stronger than that the one induced in it by $H_{a_2, k; a, A}$. The restriction of any $f \epsilon H'_{a_2, k; a, A}$ implies weak convergence in $D'(I)$.

Proof. Let ϕ be in $D(I)$ such that the support ϕ is contained in K, where K is a compact subset of I.

In virtue of the inequality

$$x^{m(2+k)/2} (A+\delta)^{-m} m^{-m\alpha} \leqslant \exp\left(\frac{a}{e} (A+\delta)^{-1/\alpha} x^{(2+k)/2\alpha}\right)$$

for every $x \epsilon I$ and $\delta > 0$, we can write

$$\| \phi \|_{a, \delta} \leqslant M_K \sum_{i=0}^{a} \sup_{x \epsilon K} \mid D^i \phi(x) \mid$$

where M_K is a positive constant depending on K. This implies our assertion.

A first characterization of the spaces $H_{a_2, k; a, A}$ is presented in the sequel.

Proposition 4. Let $\alpha > 0$. A function $\phi \epsilon C^\infty(I)$ is in $H_{a_2, k; a, A}$ if, and only if,

$$\mid (x^{-1-k} D)^q \quad (x^{a_2} \phi(x)) \mid \leqslant K_{q, \delta} \exp\left(-(a-\rho) x^{(2+k)/2\alpha}\right) \qquad \ldots(8)$$

for every $q \epsilon N$, $x \in I$ and where $K_{q, \delta}$ are suitable positive constants, $a = \frac{a}{e} A^{-1/\alpha}$ and $a - \rho = \frac{a}{e}(A+\delta)^{-1/\alpha}$, with $\delta > 0$ $(0 < \rho < a)$.

Proof. Let ϕ be in $H_{a_2, k; a, A}$. One has

$$\sup_{x \in I} \mid x^{m(2+k)/2} (x^{-1-k} D)^q (x^{a_2} \phi(x)) \mid \leqslant c_{q, \delta} (A+\delta)^m m^{m\alpha}$$

for every $m, q \in N$ and $\delta > 0$.

Therefore

$$\mid (x^{-1-k} D)^q (x^{a_2} \phi(x)) \mid \leqslant C'_{q, \delta} \exp\left(\frac{a}{e} x^{(2+k)/2\alpha} (A+\delta)^{-1/\alpha}\right) \qquad \ldots(9)$$

for every $x \in I$, because

$$\inf_{m\in N} \left[x^{-m(2+k)/2} (A+\delta)^m m^{m\alpha}\right] \leqslant M \exp\left(-\frac{a}{e} x^{(2+k)/2\alpha} (A+\delta)^{-1/\alpha}\right)$$

M being a positive constant. By setting $a-\rho=\frac{a}{e}(A+\delta)^{-1/a}$ and $a=\frac{a}{e}A^{-1/a}$ in (9) one has

$$|(x^{-1-k}D)^q (x^{\alpha_2}\phi(x))| \leqslant C'_{q,\rho} \exp(-(a-\rho)x^{(2+k)/2a}), \text{ for every}$$

$x\in I$.

Reciprocally, if ϕ is in $C^\infty(I)$ and it satisfies (8) then we can write

$$|(x^{-1-k}D)^q (x^{\alpha_2}\phi(x))| \leqslant K_{q,\delta}\, x^{-m(2+k)/2}(A+\delta)^m m^{ma},$$

for each $x\in I$, q, $m\in N$ and $\delta>0$, where $K_{q,\delta}$ are positive constants, because

$$\exp(-(a-\rho)x^{(2+k)/2\alpha}) \leqslant \min_{m\in N} x^{-m(2+k)/2}(A+\delta)^m m^{m\alpha}.$$

This completes the proof of this assertion.

REMARK 1. Let $\alpha=0$. If $\phi\in H_{\alpha_2,k;\alpha,A}$ then

$$\sup_{x\in N} |x^{m(2+k)/2}(x^{-1-k}D)^q (x^{\alpha_2}\phi(x))| \leqslant C_{q,\delta}(A+\delta)^m$$

for every $m\in N$. Hence

$$|(x^{-1-k}D)^q (x^{\alpha_2}\phi(x))| \leqslant C_{q,\delta}\min_{m\in N}\left(\frac{A+\delta}{x^{(2+k)/2}}\right)^m = \begin{cases}1, & x<A^{(2+k)/2}\\ 0, & x>A^{(2+k)/2}\end{cases}$$

and we can conclude that the support of ϕ is contained in $(0, A^{(2+k)/2})$.

Note that the space $H_{\alpha_2,k;0,A}$ reduces to the space $B_{\mu,A}$ introduced by A. H. Zemanian [19] when $\alpha_2=-\mu-\frac{1}{2}$ and $k=0$.

We now give a second characterization of the space $H_{\alpha_2,k;\alpha,A}$.

Proposition 5. Let $\alpha>0$ and $1+k$ be a non-negative integer. A function $\phi\in C^\infty(I)$ belongs to $H_{\alpha_2,k;\alpha,A}$ if, and only if, ϕ is in $H_{\alpha_2,k}$ and

$$|D^q(x^{\alpha_2}\phi(x))| \leqslant K_{q,\rho}\exp(-(a-\rho)x^{(2+k)/2a}) \qquad ...(10)$$

for every $x\in I$, $q\in N$ and $0<\rho<a$, where $a=\frac{a}{e}A^{-1/\alpha}$ and $a-\rho=\frac{a}{e}(A+\delta)^{-1/a}$ with $\delta>0$.

Proof. Let ϕ be in $H_{\alpha_2,k}$. We can insure that

$$\sup_{x\in I} |x^{m(2+k)/2}(x^{-1-k}D)^q (x^{\alpha_2}\phi(x))| \leqslant K_{m,q} \qquad ...(11)$$

for every m, $q\in N$ and $K_{m,q}$ being suitable constants. If ϕ satisfies the inequality (10) then by using (11) we get

$$|(x^{-1-k}D)^q(x^{\alpha_2}\phi(x))| \leqslant A_{q,\delta}\exp(-(a-\rho)x^{(2+k)/2\alpha}),$$

for every $x\epsilon I$, with $0<\rho<a$. Hence, in view of proposition 4, ϕ is in $H_{\alpha_2,\, \mathrm{k};\, \alpha,\, A}$.

On the other hand, if $q\in N$ and $x\in I$ then

$$D^q(x^{\alpha_2}\phi(x)) = \sum_{i=1}^{q} C_i\, x^{a(i)}(x^{-1-k}D)^i(x^{\alpha_2}\phi(x))$$

where $\alpha(i)$ are positive numbers $i=1, 2, \ldots, q$. Therefore by using again Proposition 4 we get

$$|D^q(x^{\alpha_2}\phi(x))| \leqslant K_{q,\rho}\exp(-(a-\rho)x^{(2+k)/2a})$$

for every $x\in I$ and $0<\rho<a$.

Proposition 6. If $p=(2+k)n$, with $n\in N$, then

$$H_{\alpha_2-p,\, k;\, a,\, A} \subset H_{\alpha_2,\, k;\, a,\, A}$$

and the inclusion is continuous.

Proof. Assume $p=2+k$. If $m\in N$ and $q\in N-\{0\}$ then we have

$$x^{m(2+k)/2}(x^{-1-k}D)^q(x^{\alpha_2}\phi(x)) = x^{m(2+k)/2}\{x^{2+k}(x^{-1-k}D)^q(x^{\alpha_2-2-k}\phi(x)) + (2+k)q(x^{-1-k}D)^{q-1}(x^{\alpha_2-2-k}\phi(x))\}$$

Hence, for every $\phi\in H_{\alpha_2,\, 2-k,\, k;\, a,\, A}$ one has

$$\sup_{x\in I}|x^{m(2+k)/2}(x^{-1-k}D)^q(x^{\alpha_2}\phi(x))| \leqslant C_{q,\delta}\{(A+\delta)^m m^{ma} + (A+\delta)^{m+2}(m+2)^{(m+2)a}\} \leqslant C'_{q,\eta}(A+\eta)^m m^{ma}, \quad \ldots(12)$$

for every $\eta>0$ where $C_{q,\delta}$ and $C'_{q,\eta}$ are constants, because $(2+m)^{2\alpha}\leqslant(1+\varepsilon)^m C_\varepsilon$, with $C>0$ and C_ε is a constant, and $\left(\frac{m+2}{m}\right)m^{\alpha} < e^{2\alpha}$, for every $m\in N$.

An inequality similar to (12) can be proved when $q=0$.

Therefore $H_{\alpha_2-2-k,\, k;\, a,\, A} \subset H_{\alpha_2,\, k;\, a,\, A}$ and the inclusion is continuous.

By proceeding by induction our assertion can be established.

Proposition 7. Let $0<A_1<A_2$. $H_{\alpha_2,\, k;\, a,\, A_1}$ is contained in $H_{\alpha_2,\, k;\, a,\, A_2}$ and the topology of $H_{\alpha_2,\, k;\, a,\, A_1}$ is stronger than that the one induced in it by $H_{\alpha_2,\, \mathrm{k};\, a,\, A_2}$.

To prove this it is sufficient to see that

$$\sup_{x\in I} \left| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) \right| \leqslant C_{q,\delta} (A_1+\delta)^m m^{m\alpha} \leqslant C_{q,\delta} (A_2+\delta)^m m^{m\alpha}$$

for every $\phi \in H_{\alpha_2, k; \alpha, A_1}$, $m, q \in N$ and $\delta > 0$.

The last proposition allows one to define the countably union space

$$H_{\alpha_2, \mathrm{k}; \alpha} = \bigcup_{A=1}^{\infty} H_{\alpha_2, k; \alpha, A}$$

which is equipped with the usual topology (see [20]).

2.2. The space $\mathrm{H}_{\alpha_2, k}^{\beta, B}$.

Let $\alpha_2, k \in R$, $2+k>0$, $\alpha \geqslant 0$ and $A>0$. We define a testing function space $H_{\alpha_2, k}^{\beta, B}$ as follows. A complex valued function $\phi(x)$ is in $H_{\alpha_2, k}^{\beta, B}$ if, and only if, it is defined and smooth on $0<x<\infty$ and; for each pair of non-negative integer m and q and for each positive number δ the inequality.

$$\sup_{x\in I} \left| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) \right| \leqslant C_{m,\rho} (B+\rho)^q q^{q\beta}$$

holds. Here $C_{m,\rho}$ is a constant depending on ϕ.

$H_{\alpha_2, k}^{\beta, B}$ is equipped with the topology generated by the multinorm $\Gamma_1 = \{ \| \ \|_{\rho, m} \}_{m \in N, \rho > 0}$, where $\| \ \|_{\rho, m}$ is defined by

$$\| \phi \|_{\rho, m} = \sup_{x \in I,\, q \in N} \frac{\left| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) \right|}{(B+\rho)^q q^{q\beta}}.$$

It can be easily prove that Γ_1 is equivalent to $\Gamma_2 = \{ \| \ \|_{1/n, m} \}_{n, m \in N}$. $H_{\alpha_2, k}^{\beta, B}$ is a countable normed space.

Some properties of the space $H_{\alpha_2, k}^{\beta, B}$ will be stated in the sequel.

Proposition 8. $H_{\alpha_2, k}^{\beta, B} \subset H_{\alpha_2, k}$ and the topology of $H_{\alpha_2, k}^{\beta, B}$ is stronger than that induced in it by $H_{\alpha_2, k}$. Moreover $H_{\alpha_2, k}^{\beta, B}$ is contained in $E(I)$.

Hence if ϕ is in $H^{\beta, B}_{a_2, k}$ then ϕ satisfies the requirements (*i*), (*ii*) and (*iii*) in Section 1.2.

Proposition 9. $H^{\beta, B}_{a_2, k}$ is a Frechet space.

Proposition 10. Let p $(2+k)n$ with $n \in N$. $H^{\beta, B}_{a_2 - p, k} \subset H^{\beta, B}_{a_2, k}$ and the inclusion is continuous.

Proposition 11. If $0 < B_1 < B_2$ the $H^{\beta, B_1}_{a_2, k} \subset H^{\beta, B_2}_{a_2, k}$ and the topology of $H^{\beta, B_1}_{a_2, k}$ is stronger than that induced in it by $H^{\beta, B_2}_{a_2, k}$.

Propositions 9 to 11 can be proved analogously to the corresponding ones in Section 2.1,

In virtue of Proposition 11 we can define the countable union spaces

$$H^{\beta, B}_{a_2, k} = \bigcup_{B=1}^{\infty} H^{\beta, B}_{a_2, k}$$

$H^{\beta, B}_{a_2, k}$ is endowed with the usual topology.

A useful test of convergence in $H^{\beta, B}_{a_2, k}$ is now presented.

Proposition 12. Let $\{\phi_\nu\}_{\nu=0}^{\infty}$ be a sequence in $H^{\beta, B}_{a_2, k}$. If it satisfies the following two requirements

(*a*) $\{\phi_\nu\}_{\nu=0}^{\infty}$ is uniformly bounded for every norm $\| \; \|_{\rho, m}$ with $m \in N$ and $\rho > 0$ and

(*b*) $(x^{-1-k} D)^q \, (x^{a_2} \phi(x)) \to 0$ as $\nu \to \infty$ uniformly in $(0, \varepsilon)$, for every $\varepsilon > 0$,

then $\phi_\nu \to 0$, as $\nu \to \infty$ in $H^{\beta, B}_{a_2, k}$.

Proof. Let $m \in N$, $\eta > 0$ and $0 < \rho' < \rho$. By (*a*) there exists a positive constant $C_{m,\rho'}$ such that $\| \phi_\nu \|^{\rho', m} < C_{m,\rho'}$, for each $\nu \in N$. Moreover, since $0 < \rho' < \rho$ there exists $q_0 \in N$ such that

$$\left[\frac{B+\rho'}{B+\rho}\right]^q < \frac{\eta}{C_{m,\rho'}}, \text{ for every } q \geqslant q_0.$$

Therefore, if $q \geqslant q_0$ then

$$| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) | \leqslant \eta (B+\rho)^q q^{q\beta} \qquad ...(13)$$

for every $x \in I$ anp $\nu \in N$. Let now $q \leqslant q_0$ and $x^{(2+k)/2} > \frac{C_{m+1,\rho}}{\rho}$. We have

$$| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) | \leqslant x^{-(2+k)/2} \| \phi_\nu \|^{\rho, m+1} (B+\rho)^q q^{q\beta}$$
$$< \eta (B+\rho)^q q^{q\beta}, \text{ for every } \nu \in N.$$

Finally, if $q \leqslant q_0$ and $x^{(2+k)/2} < \frac{C_{m+1,\rho}}{\rho}$ then, in view of uniformly convergence in (*b*), (13) holds provided that $\nu \geqslant \nu_0$ for some $\nu_0 \in N$.

Hence $\| \phi_\nu \|^{m,\rho} < \eta$, for every $\nu \geqslant \nu_0$ and $\phi_\nu \to 0$ as $\nu \to \infty$ in $H^{\beta, B}_{\alpha_2, k}$.

2.3. The space $H^{\beta, B}_{\alpha_2, k; \alpha, A}$.

Let $\alpha_2, k \in R$, with $2+k > 0$, $\alpha, \beta \geqslant 0$ and $A, B > 0$. The space $H^{\beta, B}_{\alpha_2, k; \alpha, A}$ consists of all functions $\phi(x)$ defined and smooth on $0 < x < \infty$ such that the number

$$\| \phi \|^{\rho}_{\delta} = \sup_{x \in I,\, m,\, q \in N} \frac{| x^{m(2+k)/2} (x^{-1-k} D)^q (x^{\alpha_2} \phi(x)) |}{(B+\rho)^q q^{q\beta} (A+\delta)^m m^{m\alpha}}$$

exists for every δ and $\rho > 0$. $H^{\beta, B}_{\alpha_2, k; \alpha, A}$ is endowed with the topology generated by the multinorm $\Gamma_1 = \{ \| \; \|^{\rho}_{\delta} \}_{\rho, \delta > 0}$. Γ_1 is a multionorm equivalent to the countably multinormed $\Gamma_2 = \{ \| \; \|^{1/n}_{1/n} \}_{n \in N}$. Hence $H^{\beta, B}_{\alpha_2, k; \alpha, A}$ is a Hausdorff locally convex linear space that satisfies the first axiom of countability.

The following topological properties of $H^{\beta, B}_{\alpha_2, k; \alpha, A}$ can be proved as the ones in Sections 2.1 and 2.2.

Proposition 13. $H^{\beta,B}_{a_2,k;a,A} \subset H_{a_2,k}$ and the inclusion is continuous.

Hence $H^{\beta,B}_{a_2,k;a,A}$ is contained in $E(I)$.

Proposition 14. $H^{\beta,\beta}_{a_2,k;a,A}$ is a Frechet space.

Proposition 15. If $p=(2+k)n$, with $n\in N$, then $H^{\beta,B}_{a_2-p,k;\alpha,A} \subset H^{\beta,B}_{a_2,k;\alpha,A}$ and the topology of $H^{\beta,B}_{a_2-p,k;\alpha,A}$ is stronger than the one induced in it by $H^{\beta,B}_{a_2,k;a,A}$.

Proposition 16. Let $\alpha>0$. A function $\phi\in C^\infty(I)$ is in $H^{\beta,B}_{a_2,k;\alpha,A}$ if, and only if,

$$|(x^{-1-k}D)^q(x^{a_2}\phi(x))| \leqslant C_{\delta,\rho}\exp\left(-(a-\rho)x^{(2+k)/2\alpha}\right)(B+\delta)^q q^{q\beta}$$

for every $x\in I$, where $a=\frac{\alpha}{e}A^{-1/\alpha}$, $a-\rho=\frac{\alpha}{e}(A+\eta)^{-1/\alpha}$, $0<\rho<a$ and $\delta,\eta>0$.

Proposition 17. If $0<A_1<A_2$ and $0<B_1<B_2$, then $H^{\beta,B_1}_{a_2,k;\alpha,A_1}$ is contained in $H^{\beta,B_2}_{a_2,k;a,A_2}$ and the inclusion is continuous.

This assertion allows one to define the countable union space

$$H^{\beta}_{a_2,k;a}=\bigcup_{A,B=1}^{\infty} H^{\beta,B}_{a_2,k;a,A}.$$

$H^{\beta}_{a_2,k;a}$ is endowed with the usual topology.

Proposition 18. Let $\{\phi_v\}_{v=0}^{\infty}$ be a sequence in $H^{\beta,B}_{a_2,k;a,A}$. If $\{\phi_v\}_{v=0}^{\infty}$ satisfies the two following conditions (*a*) $\{\phi_\nu\}_{\nu=0}^{\infty}$ is uniformly bounded for every norm $\| \ \|_{\delta}^{\rho}$,

(*b*) $(x^{-1-k}D)^q\,(x^{a_z}\phi_\nu(x))$ converges to zero, as $\nu\to\infty$, uniformly in $0<x<\varepsilon$, for every $\varepsilon>0$, $q\in N$,

then $\phi_\nu\to 0$, as $\nu\to\infty$ in $H^{\beta,\,B}_{a_2,\,k;\,a,\,A}$.

Remark 2. The spaces $H_{a_2,\,k;\,a,\,A}$, $H^{\beta,\,B}_{a_2,\,k}$ and $H^{\beta,\,B}_{a_z,\,k;\,a,\,A}$ reduces $H_{\mu,\,a,\,A}$, $H^{\beta,\,B}_{\mu}$ and $H^{\beta,\,B}_{\mu,\,a,\,A}$ introduced by W. Y. Lee [7], respectively, when $a_z=-\mu-\frac{1}{2}$ and $k=0$.

2. 4. Nontriviality of the spaces $H^{\beta,\,B}_{a_2,\,k;\,a,\,A}$

This section if devoted to study the nontriviality of the spaces $H^{\beta}_{a_z,\,k;\,a}$. Firstly we establish relations between our spaces and the spaces of type S^{β}_{a} introduced by I. M. Gelfand and G. E. Shilov [4].

Proposition 19. The operation $H:\phi\to x^{-a_2}\phi(x^{2+k})$ is a continuous linear mapping from

(*a*) $S_{a,\,A}$ into $H_{a_2,\,k;\,a,\,A}$,

(*b*) $S^{\beta,\,B}$ into $H^{\beta,\,B\,(2+k)}_{a_2,\,k}$

(*c*) $S^{\beta,\,B}_{a,\,A}$ into $H^{\beta,\,B\,(2+k)}_{a_2,\,k;\,a,\,A}$.

Proof. The mapping *H* is clearly linear.

To see (a) we take into account that if $\phi\in S_{a,\,A}$ then

$$\sup_{x\in R}\,|\,x^m D^q\phi(x)\,|\leqslant C_{q,\,\delta}\,(A+\delta)^m\, m^{m\alpha}$$

for every $m,\ q\in N$ and $\delta>0$. Hence, since

$$x^{m(2+k)/2}\,(x^{-1-k}D)^q\,(x^{a_z}\,H\phi(x))=x^{m/2}\,(2+k)^q\,D^q\phi\,(x),$$

for each $x\in I$, if $m=2p$, with $p\in N$, then

$$\sup_{x\in I}\,|\,x^{m(2+k)/2}\,(x^{-1-k}D)^q\,\,(x^{a_2}\,H\phi(x))\,|\leqslant C_{q,\,\delta}\,(A+\delta)^p\,p^{p\alpha}\leqslant$$

$$\leqslant C_{q,\,\delta}\,(A+\delta)^m\,m^{m\alpha} \qquad \text{...(14)}$$

Moreover if *m* is odd the inequality (14) also holds.

Therefore the mapping H is a continuous linear mapping of $S_{\alpha, A}$ into $H_{a_2, k; \alpha, A}$.

By using a technique similar to the one used to prove (a), the assertions (b) and (c) can be seen.

According to Proposition 19 and nontriviality of the spaces of type S (see [4], Vol. 2) we establish

Proposition 20. The following spaces are nontrivials.

(a) $H_{a_2, k; \alpha, A}$ and $H^{\beta, B}_{a_2, k}$, for each $\alpha, \beta \geqslant 0$ and $A, B > 0$,

(b) $H^{0, B}_{a_2, k; \alpha, A}$ and $H^{\beta, B}_{a_2, k; 0, A}$, for each $\alpha, \beta > 1$ and $A, B > 0$,

(c) $H^{\beta, B}_{a_2, k; \alpha, A}$ for every $A, B > 0$, provided that $\alpha + \beta > 1$,

(d) $H^{\beta, B}_{a_2, k; \alpha, A}$ provided that $\alpha + \beta = 1$ and $AB > (2+k)\gamma$, where γ is a suitable positive constant see [4], Vol. 2, pp. 233).

2. 5. Multipliers in the spaces $H^{\beta}_{a_2, k; \alpha}$

We wish to find functions $f\ (x)$ such that the operation $\phi \to f.\phi$ is a continuous linear mapping in spaces of type $H^{\beta}_{a_2, k; \alpha}$. f is called a multiplier in such spaces.

Proposition 21. Let f be C^∞ (I). If

$$|(x^{-1-k}D)^q\,(f(x))| \leqslant C_{q, \varepsilon} \exp\,(\varepsilon x^{(2+k)/2\alpha}),$$

for every $x \in I$, $\varepsilon > 0$ and $q \in N$, $C_{q, \varepsilon}$ are positive constants then f is a multiplier from $H_{a_2, k; \alpha, A}$ into itself.

Proof. To prove this assertion it is sufficient to see that

$$|(x^{-1-k}D)^q\,(x^{a_2}\ f(x)\phi(x))| \leqslant C_{q, \rho} \exp\,(-(a-\rho)\,x^{(2+k)/2\alpha}) \qquad \text{...(15)}$$

where $a = \frac{\alpha}{e} A^{-1/\alpha}$, $0 < \rho < a$ and $C_{q, \rho}$ are positive constants. By invoking Proposition 4 this statement is showed.

In a similar way we can establish.

Proposition 22. If $f(x) \in C^\infty$ (I) and

$$|(x^{-1-k}D)^q\,(f(x))| \leqslant C_q \exp\,(a_0 x^{(2+k)/2\alpha}),$$

for every $x \in I$, $q \in N$ and C_q being positive constants, then f is a multiplier from $H_{a_2, k; \alpha, A}$ into $H_{a_2, k; \alpha, A_0}$, where $A_0 = \left(\frac{\alpha}{e(a-a_0)}\right)^{\alpha}$ with $a_0 < a = \frac{\alpha}{e} A^{-1/\alpha}$.

Proposition 23. Let $f(x)\in C^\infty(I)$. If

$$|(x^{-1-k}D)^q(f(x))| \leqslant C_\varepsilon\, \varepsilon^q q^{q\beta}(1+x^{h(2+k)/2}),$$

for every $x\in I$, $q\in N$, $\epsilon>0$ and where C_ε are positive constants and $h\in N$, then f is a multiplier from $H^{\beta,B}_{\alpha_2,k}$ into itself.

Proof. Let ϕ be in $H^{\beta,B}_{\alpha_2,k}$. For every $q\in N$, $x\in I$, ρ, $\varepsilon>0$, one has

$$\sup_{x\in I}|x^{m(2+k)/2}(x^{-1-k}D)^q(x^{\alpha_2}\phi(x))| \leqslant \sum_{r=0}^{q}\binom{q}{r}C_\varepsilon\, \varepsilon^r r^{r\beta}(1+x^{h(2+k)/2}).$$

$$|x^{m(2+k)/2}(x^{-1-k}D)^{q-r}(x^{\alpha_2}\phi(x))| \leqslant C_{m,\rho}\, C_\varepsilon(B+\rho+\varepsilon)^q q^{q\beta} \leqslant$$
$$\leqslant C'_{m,\rho}(B+2\rho)^q q^{q\beta}$$

by setting $\rho=\varepsilon$. Hence $f(x)\phi(x)$ is in $H^{\beta,B}_{\alpha_2,k}$ and f is a multiplier from $H^{\beta,B}_{\alpha_2,k}$ into itself.

Proposition 24. If $f(x)\in C^\infty(I)$ and

$$|(x^{-1-k}D)^q(f(x))| \leqslant B_0{}^q q^{q\beta}(1+x^{h(2+k)/2}),$$

for every $x\in I$, $q\in N$ and B_0 is a positive constant and h is a nonnegative integer, then f is a multiplier from $H^{\beta,B}_{\alpha_2,k}$ into $H^{\beta,B+B_0}_{\alpha_2,k}$.

From the above results we can deduce the following ones.

Proposition 25. Let $f(x)$ be in $C^\infty(I)$. If

$$|(x^{-1-k}D)^q(f(x))| \leqslant C_\varepsilon\, \varepsilon^q q^{q\beta}\exp(\varepsilon x^{(2+k)/2\alpha}),$$

for every $x\in I$, $\varepsilon>0$ and $q\in N$, where C_ε is a positive constant then f is a multiplier from $H^{\beta,B}_{\alpha_2,k;\,\alpha,A}$ into itself.

Proposition 26. Let $f(x)$ be in $C^\infty(I)$. If

$$|(x^{-1-k}D)^q(f(x))| \leqslant B_0^q q^{q\beta}\exp(a_0 x^{(2+k)/2\alpha}),$$

for every $x\in I$, $q\in N$ and where B_0 is a positive constant, then f is a multiplier from $H^{\beta,B}_{\alpha_2,k;\,\alpha,A}$ into $H^{\beta,B}_{\alpha_2,k;\,\alpha,A_0}$ where $A_0=\left(\frac{\alpha}{e(a-a_0)}\right)^\alpha$, provided that $a=\frac{\alpha}{e}A^{-1/\alpha}>a_0$.

3. THE $F_{\alpha_0,\alpha_1,\alpha_2}$—TRANSFORMATION ON THE SPACES $H^{\beta}_{\alpha_2,k;\,\alpha}$

In [2] we proved that the $F_{\alpha_0,\alpha_1,\alpha_2}$–transformation is an automrphism onto $H_{\alpha_2, k}$ provided that $\alpha_0+\alpha_2<1$ and $\frac{\alpha_1-1}{2+k}\geqslant-\frac{1}{2}$. We now analyze the behaviour of the $F_{\alpha_0, \alpha_1, \alpha_2}$ —transformation on the spaces $H^{\beta}_{\alpha_2, k; a}$.

The following two operational rules that will be useful in the sequel can be proved according to (4) and (5) (see [2], Proposition 18)

$$N_{\alpha_0, \alpha_1, \alpha_2} F_{\alpha_0, \alpha_1, \alpha_2} \{\phi\}(y)=F_{\alpha_0-1-k, \alpha_1+2+k, \alpha_2-1} \{-x\phi\} (y) \qquad \ldots(16)$$

$$F_{\alpha_0-1-k, \alpha_1+2+k, \alpha_2-1} \{N_{\alpha_0, \alpha_1, \alpha_2}\phi\} (y)=-yF_{\alpha_0, \alpha_1, \alpha_2} \{\phi\}(y) \qquad \ldots(17)$$

for every $\phi\in H_{\alpha_2, k}$. Here $N_{\alpha_0, \alpha_1, \alpha_2}$ denotes the differential operator $x^{\alpha_0+\alpha_1} Dx^{\alpha_2}$.

Let ϕ be in $H_{\alpha_2, k}$ and let q and m be any pair of non-negative integers. Denote $\psi(y)=F_{\alpha_0, \alpha_1, \alpha_2} \{\phi(x)\} (y)$. According to (16) and (17), we get

$$\begin{aligned}
&(-y)^m N_{\alpha_0-q+1, \alpha_1+2(q-1), \alpha_2-q+1} \ldots N_{\alpha_0, \alpha_1, \alpha_2} \psi(y)\\
&=F_{\alpha_0-(1+k)(q+m), \alpha_1+(2+k)(q+m), \alpha_2-q-m} \{(-x)^n.\\
&\cdot N_{\alpha_0-m+1, \alpha_1+2(m-1), \alpha_2-m+1} \ldots N_{\alpha_0, \alpha_1, \alpha_2} \phi\} (y)
\end{aligned}$$

Therefore

$$\begin{aligned}
&y^{q-\alpha_2} (-1)^{m+q} y^m (y^{-1-k} D)^q (y^{\alpha_2} \psi(y))\\
&= y^{\alpha_0-\alpha_2-k(m+q)}\int_0^\infty J(xy; \alpha_2-q-m, \alpha_1+(2+k)(q+m),\\
&\qquad\qquad \alpha_0-(1+k)(q+m)\\
&\qquad \cdot x^{q+m-\alpha_2} (x^{-1-k} D)^m (y^{\alpha_2} \phi(x))\, dx
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&(-1)^{m+q} y^{m(2+k)/2} (y^{-1-k} D)^q (y^{\alpha_2} \psi(y))\\
&=\alpha_0-\frac{m+q}{2}k- q\frac{2+k}{2}.\\
&\int_0^\infty J(xy; \alpha_2-q-m, \alpha_1+(2+k)(q+m), \alpha_0-(1+k)(q+m) (xy)\\
&.x^{-\alpha_0-\alpha_2} x^{(2q+m)(2+k)/2} (x^{-1-k} D)^m (x^{\alpha_2} \phi(x))dx.
\end{aligned}$$

Hence, since in virtue of asymptotic behaviours (5) and (6) the function

$$J(x;\, a_2-q-m,\, a_1+(2+k)(q+m),\, a_0-(1+k)(q+m))x^{a_0}-\frac{m+q}{2}k-q\frac{2+k}{2}$$

is bounded on $0<z<\infty$, one has

$$|\, y^{m(2+k)/2}\,(y^{-1-k}D)^q(y^{a_2}\psi(y))\,| \leqslant$$
$$\leqslant MK_q\int_0^\infty x^{-a_0-a_2}\,|\, x^{(2q+m)(2+k)/2}(x^{-1-k}D)^m(x^{a_2}\phi(x))\,|\, dx \quad ...(18)$$

where M is a positive constant and K_q denotes $\left(\frac{2}{2+k}\right)^{q+\frac{a_1-1}{2+k}}$.

Assume now ϕ is in $H_{a_2,\, k;\, a,\, A}$. We choose $p\epsilon N$ such that $p\frac{2+k}{2}+a_2+a_0>1$. We get

$$|\, y^{m(2+k)/2}\,(x^{-1-k}D)^q\,(y^{a_2}\psi(y))\,| \leqslant C_{m,\delta}\, K_q\,((A+\delta)^{m+2q}\,(2q+m)^{(2q+m)\alpha}$$
$$+(A+\delta)^{2q+m+p}\,(2q+m+p,^{(2q+m+p)\alpha}) \leqslant C'_{m,\,\rho}\,(A^2(2e)^{2\alpha}\frac{2}{2+k}+\rho)^q\, q^{2\alpha q},$$

for every $\rho>0$, where $C_{m,\,\rho}$ and $C'_{m,\,\rho}$ are constants. Hence ψ y) is in $H_{a_2,k}^{2a,\,\frac{2}{2+k}A^2(2e)^{2\alpha}}$ and $F_{a_0,\, a_1,\, a_2}$ is a continuous linear mapping of $H_{a_2,\, k;\, a,\, A}$ into $H_{a_2,k}^{2a,\frac{2}{2+k}A^2(2e)^{2\alpha}}$.

On the other hand if $\psi\epsilon H^{\beta,\, B}_{a_2,\, k}$, by invoking (18) then

$$|\, y^{m(2+k)/2}\,(y^{-1-k}D)^q\,(y^{a_2}\psi(y))\,| \leqslant$$
$$\leqslant K_q\,(B+\rho)^m\, m^{m\beta}\,(C_{2q+m,\,\rho}+C_{2q+m+p,\rho}) \quad ...(19)$$

where p is again a nonnegative integer such that $p\frac{2+k}{2}+a_2+a_0>1$ and $C_{r,\,\rho}$ are positive constants.

Note that (19) does not imply that $\psi(y)\epsilon H_{a_2,\, k;\, \beta,\, B}$. However, if we consider the subspace of $H^{\beta,\, B}_{a_2,\, k}$ defined by

$$\hat{H}{}^{\beta,\, B}_{a_2,\, k}\; \{\phi\in H^{\beta,\, B}_{a_2,\, k} : \sup_{x\epsilon I}|\, x^{m(2+k)/2}\,(x^{-1-k}D)^q\,(x^{a_2}\phi(x)\,| < C_\rho\,(B+\rho)^q\, q^{q\beta},$$
$$m,\, q\epsilon N,\, \rho>0\}$$

then according to (19) it follows that

$$\sup_{y\in I} | y^{m(2+k)/2} (y^{-1-k} D)^q (y^{\alpha_2} \psi(y)) | \leqslant C_{\rho, q} (B+\rho)^m m^{m\beta},$$

for every $\phi \in \hat{H}{}^{\beta, B}_{\alpha_2, k}$.

Hence $\psi(y)$ is in $H_{\alpha_2, k; \beta, B}$ and $F_{\alpha_0, \alpha_1, \alpha_2}$ is a continuousmapping of $\hat{H}{}^{\beta, B}_{\alpha_2, k}$ into $H_{\alpha_2, k; \beta, B}$.

Let finally ϕ be in $H^{\beta, B}_{\alpha_2, k; \alpha, A}$. In view of (18) we can write

$$\sup_{y\in I} | (x^{m(2+k)/2} (y^{-1-k} D)^q (y^{\alpha_2} \psi(y)) | \leqslant K_q C_{\delta,\rho} (B+\rho)^m m^{m\alpha}.$$

$$\cdot((A+\delta)^{2q+m} (2q+m)^{(2q+m)\alpha} + (A+\delta)^{2q+m+p} (2q+m+p)^{2q+m+p)\alpha})$$

$$\leqslant C_{\eta,\varepsilon} (e^{\alpha} AB+\eta)^m m^{m(\alpha+\beta)} \left(\frac{2}{2+k} (2e)^{2\alpha} A^2+\varepsilon\right)^q q^{2\alpha q},$$

for every $\eta, \epsilon > 0$ (also $\delta, \rho > 0$), provided that $p\frac{2+k}{2} + \alpha_2 + \alpha_0 > 1$, Hence

[$F_{\alpha_0, \alpha_1, \alpha_2}$ is a continuous mapping from $H^{\beta, B}_{\alpha_2, k; \alpha, A}$ into $H^{2\alpha, \frac{2}{2+k}(2e)^{2\alpha} A^2}_{\alpha_2, k; \alpha+\beta, e^{\alpha} AB}$

We summarize these results by

Theorem 3. Let $2+k > 0$, $\alpha_0 + \alpha_2 < 1$ and $\frac{\alpha_1 - 1}{2+k} \geqslant -\frac{1}{2}$. $F_{\alpha_0, \alpha_1, \alpha_2}$ is a continuous linear mapping from

(*a*) $H_{\alpha_2, k; \alpha, A}$ into $H^{2\alpha, \frac{2}{2+k} A^2 (2e)^{2\alpha}}_{\alpha_2, k}$,

(*b*) $\hat{H}{}^{\beta, B}_{\alpha_2, k}$ into $H_{\alpha_2, k, \beta, B}$,

(*c*) $H^{\beta\ B}_{\alpha_2, k; \alpha, A}$ into $H^{2\alpha, \frac{2}{2+k} (2e)^{2\alpha} A^2}_{\alpha_2, k; \alpha+\beta, e^{\alpha} AB}$.

These properties can be extended to the corresponding countable union spaces.

Remark 3. The results presented in Theorm 3 reduces to the ones due to W. Y. Lee [7] when $\alpha_2 = \alpha_0 = -\mu - \frac{1}{2}$ and $\alpha_1 = 2\mu + 1$.

Remark 4. The following operational rules involving the Bessel type operator $B_{\alpha_0, \alpha_1, \alpha_2} = x^{\alpha_0} D x^{\alpha_1} D x^{\alpha_2}$ proved in [2]

$$F_{\alpha_0, \alpha_1, \alpha_2} \{B_{\alpha_0, \alpha_1, \alpha_2} \phi\} (y) = -y^{2+k} F_{\alpha_0, \alpha_1, \alpha_2} \{\phi\} (y) \quad ...(20)$$

$$B_{\alpha_0, \alpha_1, \alpha_2} F_{\alpha_0, \alpha_1, \alpha_2} \{\phi\}(y) = F_{\alpha_0, \alpha_1, \alpha_2} \{-x^{2+k}\phi\} (y) \quad ...(21)$$

for every $\phi \epsilon H_{\alpha_2, k}$, also hold when ϕ is in some space of type $H^{\beta}_{\alpha_2, k; \alpha}$, because $H^{\beta}_{\alpha_2, k; \alpha}$ is contained in $H_{\alpha_2, k}$. These operational rules will be useful in the last section of this paper.

4. THE GENERALIZED $F'_{\alpha_0, \alpha_1, \alpha_2}$ —TRANSFORMATION

Firstly, we note that if $f \epsilon H_{\alpha_2, k}$ then f defines a regular generalized function on $H'_{\alpha_0, k}$ through the equation

$$< f, \phi > = \int_0^\infty f(x)\, \phi(x)\, dx, \text{ for every } \phi \epsilon H_{\alpha_0, k}$$

provided that $\alpha_0 + \alpha_2 < 1$.

In effect, for every $\phi \epsilon H_{\alpha_0, k}$ one has

$$| <f, \phi> | \leqslant \int_0^\infty x^{-\alpha_0} | f(x) | \; x^{\alpha_0} | \phi(x) | \, dx \leqslant \sup_{x \epsilon I} | x^{\alpha_0} \phi(x) | \int^\infty x^{-\alpha_0} | f(x) | \, dx$$

and the requirments (*i*), (*ii*) and (*iii*) in section 2.1 imply that $\int_0^\infty x^{-\alpha_0} | f(x) | \, dx < \infty$.

Moreover if f and g belong to $H_{\alpha_2, k}$ and $f \neq g$ the f and g generate different regular distributions on $H'_{\alpha_0, k}$. In this sense, we can say that $H_{\alpha_2, k} \subset H'_{\alpha_0, k}$ provided that $\alpha_0 + \alpha_2 < 1$.

Hence, according to propositions 1, 8 and 13, $H^{\beta}_{\alpha_2, k; \alpha}$ is contained in $(H^{\beta}_{\alpha_2, k; \alpha})'$.

As we mentioned at the begining in [3] we defined the generalized $F'_{\alpha_0, \alpha_1, \alpha_2}$ —tranformation on $H'_{\alpha_0, k}$ as the adjoint of the classical

$F_{\alpha_2, \alpha_1 \alpha_0}$ —transformation in $H_{\alpha_0, k}$ by

$$<F'_{\alpha_0, \alpha_1, \alpha_2} f, \phi> = <f, F_{\alpha_2, \alpha_1, \alpha_0} \phi>, \ \phi \epsilon H_{\alpha_0, k}$$

(3) appears to be a generalization of the mixed Parseval's equation (2). Moreover the coventional $F_{\alpha_0, \alpha_1, \alpha_2}$ —transformation when acting on a function $f \epsilon H_{\alpha_2, k}$ is a special case of our generalized transformation. Indeed, set

$$F(y) = F_{\alpha_0, \alpha_1, \alpha_2} \{ f(x) \} \ (y) = y^{\alpha_0 - \alpha_2} \int_0^\infty J(xy; \alpha_2, \alpha_1, \alpha_0) f(x) \, dx$$

Since $z^{\alpha_0} J(z; \alpha_2, \alpha_1, \alpha_0)$ is bounded on $0 < z < \infty$ when $2+k > 0$ and $\frac{\alpha_1 - 1}{2+k} \geqslant -\frac{1}{2}$, and $\alpha_0 + \alpha_2 < 1$, $F(y)$ generates a regular distribution in $H'_{\alpha_0 k}$. Furthermore by our definition (3) we have

$$<F'_{\alpha_0, \alpha_1, \alpha_2} f, F_{\alpha_2, \alpha_1, \alpha_0} \phi> = <f, \phi> = \int_0^\infty f(x) \phi(x) \, dx, \ \phi \epsilon H_{\alpha_0, k}$$

and by invoking the Parseval's equation (2) we can write

$$\int_0^\infty f(x) \phi(x) \, dx = \int_0^\infty F(y) F_{\alpha_2, \alpha_1, \alpha_0} \{\phi\} \ (y) \, dy.$$

Therefore

$$<F'_{\alpha_0, \alpha_1, \alpha_2} f, F'_{\alpha_2, \alpha_1, \alpha_0} \phi> = <F(y), F_{\alpha_2, \alpha_1, \alpha_0} \{\phi\}(y)>$$

Thus our generalized tranrformation $F'_{\alpha_0, \alpha_1, \alpha_2} f$ is the regular generalized function corresponding to the conventional $F_{\alpha_0, \alpha_1, \alpha_2}$ transform $F(y) = F_{\alpha_0, \alpha_1, \alpha_2} \{ f \} \ (y)$.

According to Theorem 3 we can establish.

Theorem 4. Let $2+k>0, \frac{\alpha_1 - 1}{2+k} \geqslant -\frac{1}{2}$ and $\alpha_0 + \alpha_2 < 1$. The generalized. $F'_{\alpha_0, \alpha_1, \alpha_2}$ —transformation is a continuous linear mapping from

(*a*) $\left(H^{2\alpha, \frac{2}{2+k} A^2 (2e)^{2\alpha}}_{\alpha_0, k}\right)'$ into $(H_{\alpha_0, k; \alpha, A})'$,

(*b*) $(H_{\alpha_0, k; \beta, B})'$ into $(H^{\beta, B}_{\alpha_0, k;})'$. and

(c) $\left(H^{2\alpha, \frac{2}{2+k} (2e)^{2\alpha} A^2}_{\alpha_0, k; \alpha+\beta, e^{\alpha} AB}\right)'$ into $(H^{\beta, B}_{\alpha_0, k; \alpha, A})'$.

REMARK 5. Theorem 4 reduces to Theorem 4.2.1 presented by *W. Y.* Lee [7] when $a_0 = a_2 = -\mu - \frac{1}{2}$ and $a_1 = 2\mu + 1$.

REMARK 5. The operational rules (20) and (21) also hold when the transformation and the differential operator are understood in the generalized sense.

5. A CAUCHY PROBLEM INVOLVING THE BESSEL TYPE OPERATOR B_{a_0, a_1, a_2}

We now consider the Cauchy problem

$$\left.\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= P(B_{a_0, a_1, a_2, x}) \, u(x, t) \\ u(x, t_0) &= \phi_0(x) \end{aligned}\right\} \qquad \text{...(22)}$$

$0 < t < t_0 < T$, with $u(x, t)$ an unknown vector function, $u(x, t) = \{u_j(x, t)\}_{j=1}^{m}$, P is a polynomial matrix and ϕ_0 belongs to a suitable space of testing functions that will be specified later.

By applying formally the F_{a_0, a_1, a_2} —transformation to (22) we obtain the new equivalent problem (see (20))

$$\left.\begin{aligned} \frac{\partial}{\partial t} v(y, t) &= p(-y^{2+k}) \, v(y, t) \\ v(y, t_0) &= \psi_0(y) \end{aligned}\right\} \qquad \text{...(23)}$$

where $v(y, t) = F_{a_0, a_1, a_2} \{u(x, t); x \to y\}$ and $\psi_0(y) = F_{a_0, a_1, a_2} \{\phi_0)(y)$.

Solving (23) we get

$$v\ y, t) = Q(-y^{2+k}, t, t_0) \psi_0(y)$$

where $Q(s, t, t_0) = \exp((t - t_0)^P(s))$ is a formal matrix of analytic functions to be interpreted in the sense of Gelfand—Shilov ([4]).

Let now p_0 be the reduced order of the system (22) when B_{a_0, a_1, a_2} is replaced by $i \frac{\partial}{\partial x}$. Then according to [4, Vol. 3 pp. 55] we obtain

$$\| Q(s, t, t_0) \| \leqslant C_1 (1 + | s |)^{(m-1)p^0} \exp(b_0(t - t_0) | s |^{p^0})$$

Assuming that $t_0 \leqslant t \leqslant t_0 + T$ with $2^{p^0+1} b_0 T < \frac{1}{p_0} \theta^{p^0}$ we can write

$$\| Q(s, t, t_0) \| \leqslant C_2 \exp\left((2^{-p_0} \theta^{p_0} | s^{p_0} | \frac{1}{p_0} \right) \qquad \text{...(24)}$$

On the other hand, if q is a nonnegative integer then

$$(y^{-1-k} D)^q Q(-y^{2+k}, t, t_0) = (-(2+k)^q D_u Q(u, t, t_0)$$

with $u = -y^{2+k}$, and according to the Cauchy integral formula, one has

$$(y^{-1-k}\,D)^q\,Q(-y^{2+k},\,t,\,t_0)$$
$$=\frac{q!\,(-1)^q\,(2+k)^q}{2\pi R^q}\int_0^{2\pi} Q(-y^{2+k}+\mathrm{R}e^{i\xi},\,t,\,t_0)e^{-iq\xi}d\xi \qquad \ldots(25)$$

when $R>0$.

From (24) and (25) we can deduce

$$\|\,(y^{-1-k}\,D)^q\,(-y^{2+k},\,t,\,t_0)=$$
$$\leqslant C_2\,(2+k)^q\,q!R^{-q}\exp\left((2\theta)^{p_0}y^{(2+k)p_0}\,\frac{1}{p_0}\right)\exp\left((2\theta R)^{p_0}\frac{1}{p_0}\right)$$

and by setting $R=q^{1/p_0}$, we get

$$\|\,(y^{-1-k}\,D)^q\,Q(-y^{2+k},\,t,\,t_0)\,\| \qquad \ldots(26)$$
$$\leqslant C_3\left((2+k)\exp\left(\frac{1}{p_0}\,(2\theta)^{p_0}\right)\right)^q q^{q(a-p_0{}^{-1})}\quad\exp\left(\frac{1}{p_0}\,(2\theta)^{p_0}\;y^{(2+k)p_0}\right)$$

for every $a\geqslant 1$.

According to Proposition 26 and in virtue of (26) the matrix $Q(-y^{2+k},\,t,\,t_0)$ is a multiplier of $H^{a-\frac{1}{p_0},\,B}_{a_2,\,k;\,\frac{1}{2p_0},\,A}$ into $H^{a-\frac{1}{p_0},\,B_1}_{a_2,\,k;\,\frac{1}{2p_0},\,A_1}$,

where $A_1=\left[\dfrac{1}{2p_0e\,(a_1-a_0)}\right]^{1/2p_0}$, when $a_1=\dfrac{1}{2p_0e}A^{-1/2p_0}$, $q_0=\dfrac{1}{p_0}\,(2\theta)^{p_0}$ with $a_1>a_0$, and $B_1=B+(2+k)\,\exp\left(\dfrac{1}{p_0}\,(2\theta)^{p_0}\right)$, provided that these spaces are nontrivials. Said nontriviality is insured provided that the following conditions

(*a*) $a>\max\left\{1,\,\dfrac{1}{p_0}\right\}$

(*b*) $a=1,\; p_0>1$ and $\dfrac{AB}{2+k}\;2^{1/p_0}>\gamma$, (see Proposion 20),

hold.

In the sequel to simplify we will denote

$$\Phi_1=H^{a-\frac{1}{p_0},\,B}_{a_2,\,k;\,\frac{1}{2p_0},\,A}\quad\text{and}\quad\Phi_2=H^{a-\frac{1}{p_0},\,B_1}_{a_2,\,k;\,\frac{1}{2p_0},\,A_1}\,.$$

Note that $A<A_1$ and $B<B_1$. Hence $\Phi_1\subset\Phi_2$.

We wish now to spectify the space Φ of testing functions where ϕ_0 is.

According to Theorem 3 we can choose $\Phi=H^{\frac{1}{p_0}-\frac{a}{2},\,B_2}_{a_2,\,k;\,\frac{1}{2}\left(a-\frac{1}{p_0}\right),\,A_2}$ for suitable A_2 and B_2. Φ is nontrivial provided that the following conditions

(a') $a>2$ and $\frac{1}{p_0}<a<\frac{2}{p_0}$

(b') $a=2, \frac{1}{2}<p_0<1$ and $A_2B_2\ 2^{a-p_0^{-1}}\ \frac{1}{2+k}<\gamma,$

hold.

Hence if $1\leqslant a<2$ we must define Φ in another form, this one can be

$$\Phi=\{\phi\in H_{\alpha_2,\, k};\ :F_{\alpha_0,\, \alpha_1,\, \alpha_2}\{\phi\}\in\Phi_1\}$$

Then Φ is equipped with the topology generated by the multinorm $\Gamma=\{\|\ \ \|_{\delta}^{\rho*}\}_{\delta,\ \rho>0}$ where $\|\ \ \|_{\delta}^{\rho*}$ is defined by

$$\|\phi\|_{\delta}^{\rho*}=\|F_{\alpha_0,\, \alpha_1,\, \alpha_2}\phi\|_{\delta}^{\rho}, \text{ for every } \phi\in\Phi.$$

Now the nontriviality of Φ_1 implies the nontriviality of Φ.

Moreover according to Theorem 3 we get

$$F_{\alpha_0,\, \alpha_1,\, \alpha_2}\{\Phi_1\}\subset H^{\frac{1}{p_0},\, \frac{2}{2+k},\, A^2(2e)^{1/p_0}}_{\alpha_2,\, k;\ a-\frac{2}{2p_0},\, ABe^{\alpha}}=\Phi_3 \quad \text{and}$$

$$F_{\alpha_0,\, \alpha_1,\, \alpha_2}\{\Phi_2\}\subset H^{\frac{1}{p_0},\, \frac{2}{2+k}A_1^2(2e)^{1/p_0}}_{\alpha_2,\, k;\ a-\frac{1}{2p_0},\, A_1B_1e^{\alpha}}=\Phi_4$$

Also $\Phi_3\subset\Phi_4$ and the inclusion is continuous.

In virtue of the test of convergence established in Proposition 18, we can prove that

(I) $v(y, t)\rightarrow\psi_0(y)$, as $t\rightarrow t_0$, and

(II) $\frac{v(y,\, t+\triangle t)-v(y,\, t)}{\triangle t}\rightarrow P(-y^{1+k})\, v(y, t)$, as $\triangle t\rightarrow 0$

in the sense of convergence in Φ_2.

Therefore, in view of Theorm 3 and by using the operational rules (20) and (21) we can obtain

(I') $F_{\alpha_0,\, \alpha_1,\, \alpha_2}\{v(y, t);\ y\rightarrow x\}\rightarrow\Phi_0(x)$, as $t\rightarrow t_0$, and

$$(II')\quad \frac{F_{\alpha_0,\, \alpha_1,\, \alpha_2}\{v(y+\triangle y,\, t);\ y\rightarrow x\}-F_{\alpha_0,\, \alpha_1,\, \alpha_2}\{v(y,\, t);\ y\rightarrow x\}}{\triangle t}\rightarrow$$

$$\rightarrow P(B_{\alpha_0,\, \alpha_1,\, \alpha_2})\, F_{\alpha_0,\, \alpha_1,\, \alpha_2}\{v(y, t);\ y\rightarrow x\}, \text{ as } \triangle t\rightarrow 0,$$

in the sense of convergence in Φ_4.

Hence we can state the following

Theorem 5. Let $\phi_0 \in \Phi$. There exists a solution of (22) defined by

$$u(x, t) = F_{\alpha_0, \alpha_1, \alpha_2} \{\exp(P(-y^{2+k})(t-t_0)) F_{\alpha_0, \alpha_1, \alpha_2} \{\phi_0\}(y); y \to x\}$$

in the space of testing functions Φ_4.

We now consider the generalized Cauchy problem

$$\left.\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= P(B_{\alpha_0, \alpha_1, \alpha_2, x}) u(x, t) \\ u(x, 0) &= u_0(x) \end{aligned}\right\} \quad \ldots(27)$$

where $u_0 \epsilon \tau_0 = \left[H^{\frac{1}{p_0}, \frac{2}{2+k} A_1^2 (2e)^{1/p_0}}_{\alpha_0, k; a - \frac{1}{2p_0}, A_1 B_1 e^{\alpha}} \right]'$.

By applying the generalized $F'_{\alpha_0, \alpha_1, \alpha_2}$ —transformation to (27) we can obtain the new problem.

$$\left.\begin{aligned} \frac{\partial}{\partial t} v(y, t) &= p(-y^{2+k}) v(y, t) \\ v(y, 0) &= v_0(y) \end{aligned}\right\} \quad (28)$$

where

$$v_0(y) = F'_{\alpha_0, \alpha_1, \alpha_2} \{u_0(x)\}(y) \text{ and } v(y, t) = F'_{\alpha_0, \alpha_1, \alpha_2} \{u(x, t); x \to y\}.$$

A formal solution of (28) is the generalized function

$$v(y, t) = \exp(tp(-y^{2+k})) v_0(y) \epsilon \tau_1 = \left[H^{a - \frac{1}{p_0}, B}_{\alpha_0, k; \frac{1}{2p_0}, A} \right]'.$$

Hence according to (I′) and (II′) the generalized function

$$u(x, t) = F'_{\alpha_0, \alpha_1, \alpha_2} \{\exp(-y^{2+k}) t) F'_{\alpha_0, \alpha_1, \alpha_2} \{u_0\}(y); y \to x\} \epsilon \tau$$

is a solution of (27). Here τ is the dual space of a space of testing functions defined as Φ but replacing α_2 by α_0.

Theorem 6. If $u_0 \epsilon \tau_0$ then there exists a solution of (27) defined by

$$u(x, t) = F'_{\alpha_0, \alpha_1, \alpha_2} \{\exp(p(-y^{2+k}) t) F'_{\alpha_0, \alpha_1, \alpha_2} \{u_0\}(y); y \to x\} \epsilon \tau$$

ACKNOWLEDGEMENTS

The author is thankful to Prof. H. M. Srivastava of the University of Victoria for his valuable suggestions for the improvements in the paper.

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